Fibred models of dual-context type theory and Kripke-Joyal forcing

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Declaration

This thesis contains no material that has been accepted for the award of any other degree or diploma at any university. All the work in this thesis is my own, except where otherwise stated, and no generative AI tools were used in its preparation. The results in Chapter 7 have been published as the following:

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Abstract

The algebraic structures that feature in constructive, presheaf-based models of homotopy type theory have been studied in two ways: using the diagrammatic reasoning of category theory, and by reasoning with judgements in the *internal type theory*. These approaches are connected by the standard semantics for extensional type theory in a presheaf category $\mathcal{E}$, and more recently, have been explicitly related using a generalisation of Kripke-Joyal forcing semantics. An exception is the *universal uniform fibration*, a type-theoretic construction that requires the modal operator of *crisp type theory*, a fragment of Shulman's *spatial type theory*. Since this type theory is not internal to $\mathcal{E}$, the existing methods of relating the category-theoretic and type-theoretic descriptions do not immediately apply.

Towards precisely relating these two constructions of the universal uniform fibration, we begin by identifying the categories for which crisp type theory serves as an internal language. We develop a fibred version of Awodey's *natural models* to capture the language's dual-context structure of modal and non-modal variables. The intended model of crisp type theory is a specific presheaf topos with an idempotent comonad; we show that any category $\mathcal{C}$ with an idempotent comonad admits a fibred natural model of dual-context type theory. To move to a style of semantics more convenient for later applications, we show that if $\mathcal{C}$ has a classifier of a stable class of maps, it admits a fibred version of a *category with a classified stable class of maps* that models the dual-context structure.

Next, we specialise to the intended model arising from a presheaf category $\mathcal{E}$ with a particular idempotent comonad $\flat$. We specify how $(\mathcal{E}, \flat)$ determines a fibred category with a classified stable class of maps and show that this language validates rules of crisp type theory, notably those for *crisp* Π-*types*. We develop Kripke-Joyal forcing semantics for this internal crisp type theory. Finally, we return to the motivating problem, using the understanding of crisp type theory as an internal language to precisely relate the category-theoretic and type-theoretic versions of the construction of a universal uniform fibration.

In addition to this main project, the thesis includes distinct work on the hyperdoctrine semantics of quantified modal logic. Lawvere hyperdoctrines give categorical semantics for intuitionistic predicate logic but are flexible enough to be applied to other logics and extended to higher-order systems. We return to Ghilardi's hyperdoctrine semantics for first-order modal logic and extend it in two directions—to weaker, non-normal modal logics and to higher-order modal logics. We also relate S4 modal hyperdoctrines to intuitionistic hyperdoctrines via a hyperdoctrinal version of the Gödel-McKinsey-Tarski translation.

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1 Introduction

This thesis comprises two research projects in categorical logic. The main project is a study of crisp type theory motivated by the relationship between diagrammatic and type-theoretic ways of reasoning in a presheaf category. The background for this is provided in Sections 1.1 and 1.2. The aims of the project are set out in Section 1.3, along with the approach taken to address them. This is placed in the context of related work in Section 1.4. Section 1.5 introduces the second, smaller project of extending hyperdoctrine semantics for quantified modal logic. Finally, Section 1.6 gives an overview of the chapters of the thesis, highlighting the original contributions therein.

1.1 Presheaf models of homotopy type theory

Homotopy type theory is a young field of mathematics building on connections between Martin-Löf type theory [NPS00] and Quillen's homotopical algebra [Qui67]. It developed from studying models of identity types, beginning with the groupoid interpretation of type theory [HS96] and subsequently the connection between identity types and weak factorisation systems (wfs's for short) [GG08; AW09]. At the same time, homotopical interpretations of other type constructors were discovered [Voe06], and a complete homotopical picture of type theory lead to consequential new concepts such as the univalence axiom and higher inductive types. Research today investigates these consequences, formalising diverse parts of mathematics in this new foundation and developing computer proof-assistant tools to aid the process. Amongst this, the semantic investigations from which the field emerged remain central. Informally, a *semantics* is a precise relationship between a formal language (such as a type theory) and another mathematical setting given by a category with some structure. The language receives an *interpretation* in such a category, which is then said to *admit a model* of the theory.

Presheaf categories are a source of models for homotopy type theory due to the existence of many wfs's [Cis06]. This includes Voevodsky's simplicial set model [KL21] and subsequent models in cubical sets, which use a refinement of wfs's called *algebraic weak factorisation systems* (awfs's for short) to obtain constructive models. The theory of awfs's, studied in [Awo18; GS17; Sat17; Swa16; Swa18; BF22], is more complex because the algebraic structure on a map is not necessarily unique and so must be carried around explicitly. To manage this, it was suggested in [Coq15], and later realised in [OP18], that these models be studied using an *internal type theory*, allowing more convenient manipulation of the complex structure and facilitating formalisation in proof assistants.

The internal type theory of a presheaf category [Mai05; Pit00; See84] is an extensional dependent type theory used to reason about the category's basic structure—and its locally cartesian closed structure—in a logical fashion, with type-theoretic judgements rather than diagrams. The two different ways of reasoning are related via the standard categorical semantics for dependent type theory in presheaf categories [See84], however, the iterated dependency involved in describing awfs's make this a complex task. To improve the situation, Awodey, Gambino and Hazratpour [AGH24] developed a tool for relating these ways of reasoning, using it to reconcile the category-theoretic and type-theoretic descriptions of the awfs's considered in models of homotopy type theory. They do this by extending the technique of Kripke-Joyal forcing [LM92] originally developed for the Mitchell-Bénabou language internal to an elementary topos. They also give type-theoretic constructions of classifiers for certain types with higher structure, serving a similar purpose as standard type universes.

While [AGH24] reconcile the two styles of descriptions for a *uniform fibration*, part of an awfs, they note that their construction of a classifying universal uniform fibration involves a mix of category theory and the internal language, and cannot be done purely in the internal language. This is because there can be no version of the universal uniform fibration in the standard internal type theory of a presheaf category [OP18, Remark 7.5], since the adjunction involved in the construction cannot be internalised to this type theory. Licata et al. [LOPS18] show that a universal uniform fibration can be constructed in *crisp type theory*, a dependent type theory with a modality. However, since the

Kripke-Joyal forcing of [AGH24] is developed for the standard extensional type theory internal to a presheaf category, it does not immediately apply to crisp type theory. Furthermore, crisp type theory as used in [LOPS18] is not presented as the internal language of a category, constituting another barrier to relating it to the diagrammatic description.

Thus it remains to precisely relate category-theoretic and type-theoretic constructions of the universal uniform fibration. This forms the motivating problem for this thesis. It requires undersanding crisp type theory as the internal language of a category, and then using either its interpretation in this category or a version of Kripke-Joyal forcing for crisp type theory to relate the two constructions. We now consider to what extent the background work exists to complete the task, introducing crisp type theory and the state of understanding of its models.

1.2 CRISP TYPE THEORY AND ITS MODELS

Crisp type theory refers to the fragment of Shulman's spatial type theory [Shu18] singled-out in [LOPS18] for the problem of constructing an internal version of the universal uniform fibration. Spatial type theory features two modalities, $\sharp$ and $\flat$, corresponding respectively to the possibility and necessity modalities of modal logic. They add a notion of *cohesion* to homotopy type theory, first explored in [SS14], that enables the recovery of topological information otherwise lost due to the synthetic nature of topological notions in homotopy type theory. The language is motivated by intended models in *local toposes*, which feature two adjoint functors corresponding to $\sharp$ and $\flat$ respectively. Crisp type theory includes only the $\flat$ modality.

While there are different styles of syntax for modal type theory, spatial type theory is a dependent version of Pfenning and Davies' modal type theory [PD01], which uses a *dual-context structure*. The first context zone contains *modal* or *crisp variables* and the second context zone contains regular variables, capturing the requirement that crisp variables can only depend on other crisp variables. In this way, the modality is baked into the context structure. Figure 1.1 presents the rules of crisp type theory from [LOPS18]. Missing from the figure but also included in the language are β and η judgemental equalities, as well as crisp induction for identity types, omitted here as it is not required for the fragment of the construction of the universal uniform fibration that we consider. Note that these rules do not include the rules for the $\flat$ modality, as these are not required for the universal uniform fibration construction. Figure 1.2 augments these rules with crisp type theory versions of standard dependent sum and product types. We include these to compare them to their crisp counterparts.

By construction of the type theory with specific models in mind, models of spatial type theory—and thus crisp type theory—should be admitted in any local topos. The defining condition of a local topos is that the adjunction between the global sections and constant presheaf functors, $p^* \dashv p_*$, has a further right adjoint, $p^!$. The modalities $\sharp$ and $\flat$ are then intended to be modelled respectively by the monad $p^! p_*$ and the comonad $p^* p_*$, which we note is idempotent. Shulman remarks that this is only conjectural:

> "What needs to be done to make it precise is (1) define an appropriate sort of 'category with families' or 'contextual category' to handle our two-context type theory with crisp

CRISP VARIABLE

$$\frac{}{\Delta, x :: \sigma, \Delta' \mid \Gamma \vdash x : \sigma}$$

CRISP SUBSTITUTION

$$\frac{\Delta \mid \cdot \vdash s : \sigma \qquad \Delta, x :: \sigma, \Delta' \mid \Gamma \vdash t : \tau}{\Delta, \Delta'[s/x] \mid \Gamma[s/x] \vdash t[s/x] : \tau[s/x]}$$

CRISP Π-FORMATION

$$\frac{\Delta \mid \cdot \vdash \sigma \ \mathsf{type} \qquad \Delta, x :: \sigma \mid \Gamma \vdash \tau \ \mathsf{type}}{\Delta \mid \Gamma \vdash \Pi_{x::\sigma}\tau \ \mathsf{type}}$$

CRISP Π-INTRODUCTION

$$\frac{\Delta, x :: \sigma \mid \Gamma \vdash t : \tau}{\Delta \mid \Gamma \vdash \lambda x :: \sigma . t : \Pi_{x::\sigma}\tau}$$

CRISP Π-ELIMINATION

$$\frac{\Delta \mid \Gamma \vdash f : \Pi_{x::\sigma}\tau \qquad \Delta \mid \cdot \vdash s : \sigma}{\Delta \mid \Gamma \vdash f\, s : \tau[s/x]}$$

FIGURE 1.1: Crisp type theory

Σ-FORMATION

$$\frac{\Delta \mid \Gamma \vdash \sigma \ \mathsf{type} \qquad \Delta \mid \Gamma, x : \sigma \vdash \tau \ \mathsf{type}}{\Delta \mid \Gamma \vdash \Sigma_{x:\sigma}\tau \ \mathsf{type}}$$

Σ-INTRODUCTION

$$\frac{\Delta \mid \Gamma \vdash s : \sigma \qquad \Delta \mid \Gamma \vdash t : \tau[s/x]}{\Delta \mid \Gamma \vdash (s, t) : \Sigma_{x:\sigma}\tau}$$

Σ-ELIMINATION-1

$$\frac{\Delta \mid \Gamma \vdash f : \Sigma_{x:\sigma}\tau}{\Delta \mid \Gamma \vdash \pi_1(f) : \sigma}$$

Σ-ELIMINATION-2

$$\frac{\Delta \mid \Gamma \vdash f : \Sigma_{x:\sigma}\tau}{\Delta \mid \Gamma \vdash \pi_2(f) : \tau(\pi_1(f))}$$

Π-FORMATION

$$\frac{\Delta \mid \Gamma \vdash \sigma \ \mathsf{type} \qquad \Delta \mid \Gamma, x : \sigma \vdash \tau \ \mathsf{type}}{\Delta \mid \Gamma \vdash \Pi_{x:\sigma}\tau \ \mathsf{type}}$$

Π-INTRODUCTION

$$\frac{\Delta \mid \Gamma, x : \sigma \vdash t : \tau}{\Delta \mid \Gamma \vdash \lambda x : \sigma . t : \Pi_{x:\sigma}\tau}$$

Π-ELIMINATION

$$\frac{\Delta \mid \Gamma \vdash f : \Pi_{x:\sigma}\tau \qquad \Delta \mid \Gamma \vdash s : \sigma}{\Delta \mid \Gamma \vdash f\, s : \tau[s/x]}$$

FIGURE 1.2: Standard dependent sum and product types in crisp type theory

> variables and formulate $\sharp$ and $\flat$ as algebraic structure on such a gadget, (2) prove that syntax yields an initial one of these, and (3) construct such algebraic objects from geometric morphisms." [Shu18, Remark 7.5]

Returning to our motivating problem, although the construction of the universal uniform fibration in [LOPS18] is described as internal, the language is not explicitly presented as an internal language of a category. It is understood that this can be done, by reference to its intended models in local toposes, but the details are not developed. Thus to relate the diagammatic and type-theoretic constructions of the universal uniform fibration, we must first understand how to extract crisp type theory as the internal language of a category. This requires understanding precisely *how* a local topos and its idempotent comonad $\flat$ models the type theory.

1.3 AIMS AND APPROACH

We now summarise the aims of this research and present our approach to addressing them. The goal is to extract crisp type theory as the internal language of a category and use this to relate category-theoretic and type-theoretic descriptions of the universal uniform fibration. This requires understanding models of crisp type theory in local toposes. Since crisp type theory is motivated by the specific local topos of *De Morgan cubical sets* [CCHM18], we will investigate models in a presheaf category over a small category $\mathbb{C}$ with a terminal object, meaning $\widehat{\mathbb{C}}$ is a local topos.

To understand models of crisp type theory, we abstract to the more general case of dual-context type theories. By considering each feature of this context structure in turn, and the minimal requirements to model them, we address the first part of [Shu18, Remark 7.5]. That is, we define an appropriate sort of 'category with families' that is a fibred version of Awodey's *natural models* [Awo16]. Briefly, this involves equipping two categories with the structure to model dependent type theory—one modelling the first context zone and the other modelling the entire context—and then specifying the appropriate relationship between these type theories. The partial syntax we consider for a dual-context type theory is given in Figure 1.3.

With this understanding we can proceed to the goal of extracting crisp type theory as an internal language. We progressively zoom in on the intended model. We show first that a category with an idempotent comonad admits a fibred natural model of dual-context type theory, and then specialise this to the intended model of a presheaf category and its specific idempotent comonad. We then use this model to extract an internal dual-context type theory and validate the rules of crisp type theory. Interpreting this internal crisp type theory in the category from which it was extracted then allows us to precisely relate the category-theoretic and type-theoretic versions of the universal uniform fibration. As another tool for relating these two ways of reasoning, we develop Kripke-Joyal forcing semantics for this internal crisp type theory.

$$\textsc{ctx/emp}\ \frac{\Delta\ \mathsf{ctx}}{\Delta \mid \cdot\ \mathsf{ctx}} \qquad \textsc{ctx/ext/1st}\ \frac{\Delta \mid \cdot \vdash \sigma\ \mathsf{type}}{\Delta, x :: \sigma \mid \cdot\ \mathsf{ctx}} \qquad \textsc{ctx/ext/2nd}\ \frac{\Delta \mid \Gamma \vdash \sigma\ \mathsf{type}}{\Delta \mid \Gamma, x : \sigma\ \mathsf{ctx}}$$

$$\textsc{ctx/wk/1st}\ \frac{\Delta \mid \Gamma \vdash \tau\ \mathsf{type} \qquad \Delta \mid \cdot \vdash \sigma\ \mathsf{type} \qquad q_\sigma : \Delta.\sigma \to \Delta}{\Delta, x :: \sigma \mid \Gamma[q_\sigma] \vdash \tau[q_\sigma]\ \mathsf{type}}$$

FIGURE 1.3: Dual-context type theory

1.4 RELATED WORK

On commencing this research, existing work on modelling crisp type theory was limited to the conjecture in [Shu18, Remark 7.5], described also in [LOPS18, Remark 4.1]. Broadening the scope to models of similar modal type theories, Ritter and de Paiva [PR16] provide a fibrational semantics for a dependent modal type theory with dual-context syntax called *dependent dual intuitionistic modal* *logic*. This language differs from crisp type theory in that types depend only on modal variables, rather than both modal and standard variables. It uses as a base model of dependent type theory a variant of Ehrhard's *D-categories* [Ehr88], which is some distance from the style of model we require for our motivating problem. We note also that in this model, dependent sum types are required to model the modality, whereas we choose to develop a model in a more modular fashion. Nevertheless, we acknowledge the influence of this work on our understanding of dependent modal type theory.

We note other related work that existed at the beginning of this project, but which had less influence due to a combination of differing motivation, style of syntax, and flavour of categorical model. Kavvos [Kav20] gives an extensive treatment of dual-context type theories, including their metatheory and semantics in a cartesian closed category with a product-preserving endofunctor. Birkedal et al. [BCM+20] extend the *category with families* model of dependent type theory [Dyb95; Hof97] with the notion of *dependent right adjoints* to provide semantics for a modal type theory with Fitch-style syntax, rather than a dual-context system. A prevailing theme of recent work on modal type theory, sparked by the addition of modalities to homotopy type theory, is the process of going from a semantic situation to a well-behaved type theory. This is illustrated by the question of how best to add modalities with the property of being a comonad and a monad that are themselves adjoint, as in $\flat$ and $\sharp$. This is the concern of *mode theory* [LS16; LSR17] and the subsequent dependently-typed general frameworks of Multimodal Type Theory [GKNB21] and Fitch Type Theory [GCK+22]. This work includes investigations of both syntax and semantics, where the syntax is again Fitch-style. These are combined into a general modal type theory, Multimodal Adjoint Type Theory, in [Shu23]. The semantics in [GKNB21] is given by *modal natural models*, later generalised for the aforementioned Multimodal Adjoint Type Theory with *adjoint modal natural models*. While this work is related, our substantially different motivation—to extract the modal type theory as an internal language of a category—leads to a distinct technical development.

Some work emerged during the course of this thesis with closer connections. Zwanziger [Zwa23] develops the notions of *natural display topos* and *natural cartesian display comonad*, and uses these to sketch an interpretation of a comonadic dependent type theory akin to crisp type theory. While this uses the same style of categorical model as we do, we do not require the full strength of this development for our purposes. For example, we are free to assume that the comonad in our model is idempotent, whereas this assumption is not required by Zwanziger. Gratzer's extensive work on the syntax and semantics of modal type theory [Gra23] focuses on Fitch-style syntax, but includes a discussion of the semantics of dual-context type theory. This reaches the same conclusions as we do in Chapter 2, and includes a definition of model of *dual-context type theory without modal types* [Gra23, pg.77] that is similar to our definition of *fibred natural model* (Definition 2.3.1). Despite this overlap, the results of Chapter 2 are developed and used in the rest of thesis in a different way to the work of Gratzer.

1.5 Hyperdoctrine semantics for quantified modal logic

We now introduce the second project of the thesis. Traditional Kripke semantics for propositional modal logic do not automatically extend to the first-order case, with several instances of well-motivated but incomplete extensions of Kripke-complete propositional logics [Cre95]. Turning to alternative semantics, category-theoretic methods have been used extensively by Ghilardi and Meloni [GM88; GM90; Ghi91; GM91; Ghi01] for mathematical and philosophical investigations of quantified modal logic beyond the reach of Kripke semantics.

Amongst the category-theoretic tools deployed are Lawvere's *hyperdoctrines* [Law69]. Hyperdoctrines provide semantics for first-order logics that reduce to familiar algebraic semantics on the propositional level. Originally conceived for intuitionistic predicate logic, they are flexible enough to be applied to other logics and extended to higher-order systems. Hyperdoctrine semantics for first-order normal modal logics are presented in [BG07], where they are used by Ghilardi as a unifying tool for studying other non-Kripkean modal semantics, while Awodey, Kishida and Kotzsch [AKK14; Kot16] provide topos-theoretic hyperdoctrine semantics for higher-order modal logic based on intuitionistic S4.

We make three contributions to modal hyperdoctrine. The first is a very general presentation. Ghilardi's presentation in [BG07] concerns a single-sorted typed language and a base propositional modal logic of S4, while advising that it is straightforward to generalise. We follow this guidance to present modal hyperdoctrine semantics for a many-sorted typed language and a base propositional modal logic of the weaker *non-normal* class. The second contribution is to connect modal hyperdoctrines—in the case of S4 modal logics—to intuitionstic hyperdoctrines via a translation theorem. The third is to define higher-order modal hyperdoctrines for non-normal modal logics and prove their soundness and completeness. This complements the aforementioned work of Awodey et al., in which the topos-theoretic nature of their semantics prohibits generalising to bases weaker than S4. This project has been published as [VM24] and contributes one chapter of the thesis.

1.6 Overview

Chapter 2: Fibred models of dual-context type theory

Chapter 2 introduces the two kinds of categorical models for dependent type theory that will be used throughout this thesis: *categories with stable maps and a classifier* [Tay87; KL21], and *natural models* [AGH24]. We recall results from [AGH24] describing the relationship between stable maps in the former model and *represented pullbacks* in the latter. We extend this relationship to the models' respective notions of type universe, specifying when the *representable natural transformation* of a natural model gives rise to a classifier in the sense of the first model. We then consider the context rules for dual-context type theory to build, in a modular fashion, a definition of *fibred natural model of dual-context type theory* on a functor $P : \mathcal{E} \to \mathcal{B}$. We conclude with an alternative axiomatisation of this definition, useful for Chapter 3.

Chapter 3: Fibred models from a category with an idempotent comonad

Towards the intended models of crisp type theory, we show that *any* category $\mathcal{C}$ with an idempotent comonad admits a fibred natural model of dual-context type theory. We also show that if there is a classifier for the stable maps/represented pullbacks in $\mathcal{C}$, in the sense of our other style of model, then there are classifiers for the stable maps/represented pullbacks in the fibred natural model. Thus we have a fibred version of *category with stable maps and a classifier*. This alternative semantic setting will be used for the applications in Chapters 4-6.

Chapter 4: Presheaf-based models of crisp type theory

The results of Chapters 2 and 3 now allow us to return to the original goal of extracting crisp type theory as the internal language of a category. We specialise the result from Chapter 3 to a presheaf category $\widehat{\mathbb{C}}$ and the idempotent comonad $\flat$, showing that it admits a fibred natural model and a fibred version of a category with stable maps and a classifier. We then prove that the total category $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ of this fibred model is locally cartesian closed, and motivated by the type theory we intend to extract, consider adjoints to pullback along the vertical and horizontal small maps that model the two kinds of context extension in a dual-context type theory. Next, we introduce an internal language to this category, and prove that judgements in this language validate the context rules of dual-context type theory. We also prove that the language supports standard dependent sum and product types, as well as the crisp product types used in [LOPS18] for the type-theoretic construction of the universal uniform fibration. Finally, we show that crisp type theory is a subtheory of this internal language.

Chapter 5: Kripke-Joyal semantics for crisp type theory

Having presented crisp type theory as an internal language, we now set up the technique of Kripke-Joyal forcing in this setting, following [AGH24]. Firstly, we prove that there are *canonical generators* for the objects of the total category $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ of the fibred model from Chapter 4. That is, we give a result analogous to that for a presheaf category which says that every object is isomorphic to a colimit of representables. We then define the forcing condition and prove its basic properties.

Chapter 6: Universal uniform fibration in presheaf-based models

In the final chapter of the main project, we use our understanding of crisp type theory as an internal language to relate part of the category-theoretic and type-theoretic constructions of the universal uniform fibration. First, we recall from [OP18; LOPS18] why a universal uniform fibration cannot be constructed in the standard internal type theory of a presheaf category. Then we give a category-theoretic construction of the universal uniform fibration, synthesising those in [LOPS18] and [AGH24]. Finally, we precisely relate parts of this proof to the type-theoretic one from [LOPS18]. In this way, we improve or expand on the two existing proofs: compared to [LOPS18], we have a notion of model, and compared to [AGH24], we are not limited by working in the ordinary extensional type theory of a presheaf category.

Chapter 7: Hyperdoctrine semantics for quantified modal logic

We begin by introducing hyperdoctrine semantics for intuitionistic logic and the syntax of a particular weak class of first-order modal logic called *non-normal* first-order modal logic. Where [BG07] give hyperdoctrine semantics for first-order, *normal* modal logic, we show that they also provide semantics for this weaker, non-normal class by proving soundness and completeness. Then we connect modal hyperdoctrines—in the case of S4 modal logics—to intuitionstic hyperdoctrines via a translation theorem. Finally, we define higher-order modal hyperdoctrines for non-normal modal logics and prove their soundness and completeness.

Chapter 8: Conclusion

As well as summarising the developments of the thesis, discussion of future research directions can be found here.

2 Fibred models of dual-context type theory

Introduction

For our motivating problem, we wish to extract crisp type theory as the internal language of a category. Standard extensional dependent type theory is understood as the internal language of locally cartesian closed categories using the fact that a model of dependent type theory can be constructed from such categories [See84; Hof94]. Categorical models of dependent type theory are variable-free presentations of the basic framework of dependent types, namely contexts, substitutions, types, terms, and context extension. Since contexts in crisp type theory behave differently to those in standard type theory, their models do not immediately apply. In this chapter, we show how the dual-context structure of crisp type theory can be modelled by considering not just a single category, but two related categories. That is, we equip two categories with the structure to model dependent type theory—one modelling the first context zone and the other modelling the entire context—and then specify the appropriate relationship between these type theories for capturing dual-context type theory.

In Section 2.1, we recall two ways of modelling the core of dependent type theory, namely *categories with a classified stable class of maps* [Tay87; KL21] and *natural models* [Awo16]. Both flavours of model feature a notion of *type universe* to solve the coherence problem arising from the strictly functorial nature of substitution in type theory. In addition to recalling the relationship between these models from [Awo16], we specify how these two notions of type universe are related (Proposition 2.1.22), which to the best of our knowledge does not already appear in the literature. We use a presheaf category as a running example for both models, presheaves being locally cartesian closed categories and the setting of our later applications.

After recalling some basic notions from fibred category theory, in Section 2.2 we consider the features of a dual-context structure that differ from standard contexts. In a modular fashion, we take a dual-context rule and provide a categorical analysis, explaining how the semantics works and how it corresponds to an expression in an internal language. This culminates in the definition of *fibred natural model of dual-context type theory* (Definition 2.3.1) in Section 2.3. We also prove an alternative axiomatisation of this definition (Proposition 2.3.8), which will be used in Chapter 3 when we show that a category with an idempotent comonad admits a fibred natural model.

2.1 PRELIMINARIES: CATEGORICAL MODELS OF DEPENDENT TYPE THEORY

We are interested in two different notions of categorical model of dependent type theory: the first is a *natural model* on a category [Awo16] and the second is a category with *stable maps and a classifier* [Tay87; KL21]. [1] The former will be used in Section 2.2 to develop a model for dual-context type theory as a *fibred natural model* on a functor, while the latter is the setting of the motivating problem from [AGH24] and so is useful for applications in later chapters. Hence, after introducing the definitions of these models, we spell out their connection so that we can move between them at our convenience. We also include some necessary preliminaries on fibred category theory.

We begin with some basic conventions to be followed throughout the thesis. We work assuming a fixed inaccessible cardinal $\varkappa$ and introduce the following terminology regarding size.

2.1.1. *Terminology.* (a) A set is *$\varkappa$-small* if it has cardinality less than $\varkappa$.

(b) A category is *locally small* if its hom-classes are sets, and *locally $\varkappa$-small* if its hom-classes are $\varkappa$-small sets.

(c) A category is *small* if its class of objects is a set and it is locally small, and *$\varkappa$-small* if its class of objects is $\varkappa$-small and it is locally $\varkappa$-small.

(d) The category of sets and functions is denoted Set, the category of pointed sets and point-preserving functions is denoted $\mathsf{Set}^{\bullet}$.

(e) The category of $\varkappa$-small sets and functions is denoted $\mathsf{Set}_{\varkappa}$ and the category of pointed $\varkappa$-small sets and point-preserving functions is denoted $\mathsf{Set}^{\bullet}_{\varkappa}$

We will be working with the category of presheaves over a small category $\mathcal{C}$, denoted $\widehat{\mathcal{C}}$. We write $\text{よ} : \mathcal{C} \to \widehat{\mathcal{C}}$ for the Yoneda embedding and adopt the following conventions when working with a presheaf category.

2.1.2. *Conventions.* (a) We will use the Yoneda embedding to identify objects and arrows in $\mathcal{C}$ with representable presheaves and natural transformations between them, using the same symbol to denote both an object in $\mathcal{C}$ and the representable presheaf associated to it.

(b) Given a presheaf U and an object X in $\mathcal{C}$, we use the same name for an element $x \in U(X)$ and the corresponding natural transformation $x : X \to U$, as permitted by the Yoneda lemma.

Category with stable maps and a classifier

Let $\mathcal{C}$ be a small category. We single out a special class of maps to capture dependent types in context.

2.1.3. DEFINITION (stable class of maps). A class S of morphisms in $\mathcal{C}$, typically denoted with '$\twoheadrightarrow$', is called a *stable class* if the following conditions are satisfied:

(i) for every map $f : C \twoheadrightarrow D$ in S, pullback along any map in $\mathcal{C}$ exists

[1] This is a version of *display map category* [Tay87] augmented with a classifer by Voevodksy [KL21]. For an example of this style of model in use, see [AGH24].

(ii) given any pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ B \arrow[r] & D, \end{tikzcd} \tag{2.1}$$

if f is in S then g is in S.

2.1.4. *Remark.* Informally, we can see the connection between stable maps and dependent types via the *syntactic category* or *category of contexts* (see, for example, [Jac99, Section 10.1]). This is a category whose objects are contexts, that is, lists of variable-type pairs, and whose morphisms are substitutions. Amongst these morphisms are *dependent projection maps*, which take a context

$$x_1 : \sigma_1, x_2 : \sigma_2(x_1), \dots, x_n : \sigma_n(x_1, \dots, x_{n-1}), y : \tau(x_1, \dots, x_n)$$

to the context

$$x_1 : \sigma_1, x_2 : \sigma_2(x_1), \dots, x_n : \sigma_n(x_1, \dots, x_{n-1}),$$

and so capture the projected-away type τ. These morphisms form a stable class of maps in the category of contexts.

2.1.5. EXAMPLE (Small maps in a presheaf category). The category of presheaves $\widehat{\mathbb{C}}$ over a small category $\mathbb{C}$ has a stable class of maps given by the *small maps*, denoted $\mathcal{S}$. A map $p : A \to X$ in $\widehat{\mathbb{C}}$ is small if for every c in $\mathbb{C}$ and $x \in X(c)$, the presheaf A_x obtained by pullback

$$\begin{tikzcd} A_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ よc \arrow[r, "x"'] & X \end{tikzcd}$$

is small, that is, all its values are κ-small sets. To see that small maps form a stable class, we first note that presheaf categories have all limits, so pullback of a small map along any map exists. Now let $p : A \to X$ be small and consider any pullback square

$$\begin{tikzcd} B \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

in $\widehat{\mathbb{C}}$. To see that q must be small, let c be an object in $\mathbb{C}$ and y an element of $Y(c)$, and consider the pullback of q along y:

$$\begin{tikzcd} B_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ よc \arrow[r, "y"'] & Y \arrow[r, "f"'] & X. \end{tikzcd}$$

Since p is small, the presheaf obtained by pullback along $f \circ y : よc \to X$ is small, and by the essential uniqueness of pullbacks, is isomorphic to B_y. Therefore, q is small.

2.1.6. *Observation.* Stable classes of maps are closed under isomorphism, where the notion of isomorphism comes from viewing maps as objects in the arrow category $\mathcal{C}^{\to}$.

The original interpretation of dependent type theory in locally cartesian closed categories [See84] models the strictly functorial operation of substitution into types and terms by the pseudofunctorial operation of pullback. Solutions to this coherence problem followed, notably [Cur90] and [Hof94]. A more recent approach due to Voevodsky (see, e.g., [KL21]) models substitution into terms by the strictly functorial operation of composition by using a *classifier* for the stable maps, as follows.

2.1.7. DEFINITION (category with stable maps and a classifier). A *category with stable maps and a classifier* consists of a category $\mathcal{C}$, a stable class of maps $\mathsf{S} \subseteq \mathsf{mor}(\mathcal{C})$, and a map $\pi : \widetilde{V} \twoheadrightarrow V$ in S such that for every $p : D \twoheadrightarrow C$ in S there exist maps f and g in $\mathcal{C}$ making the following square into a pullback:

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{V} \arrow[d, "\pi"] \\ C \arrow[r, "f"'] & V. \end{tikzcd} \tag{2.2}$$

2.1.8. *Remark.* The notion of a classifier $\pi : \widetilde{V} \twoheadrightarrow V$ may be regarded in two ways: as a property of the map π, meaning that for each map $p : D \twoheadrightarrow C$ in S, one can find maps $f : C \to V$ and $g : D \to \widetilde{V}$ fitting into the pullback in (2.2); or as structure on the map π, meaning there is an explicitly given function that specifies a choice of such maps f and g for every map $p : D \twoheadrightarrow C$.

Generally, we will work with the property version of a classifer, but for instances where the structured version is required, we introduce the following terminology.

2.1.9. *Terminology.* Let S be a stable class of maps in $\mathcal{C}$ and let $\pi : \widetilde{V} \twoheadrightarrow V$ be a classifier for S. Viewing the classifier as structure, we introduce the following terminology.

(i) The specified choice of map $f : C \to V$ in (2.2) is called the *classifying map* for $p : D \twoheadrightarrow C$, denoted $\chi_p : C \to V$. We say that p is *classified* by χ_p.

(ii) The specified choice of pullback of π along χ_p is denoted

$$\begin{tikzcd} C.\chi_p \arrow[r, "q_{\chi_p}"] \arrow[d, "p_{\chi_p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{V} \arrow[d, "\pi"] \\ C \arrow[r, "\chi_p"'] & V. \end{tikzcd}$$

We call $C.\chi_p$ the *comprehension of χ_p with respect to V* and p_{χ_p} the *display map* associated to χ_p.

(iii) It follows that every stable map is isomorphic to a display map, so for every stable map $p : D \twoheadrightarrow C$, there is a diagram

$$\begin{tikzcd} D \arrow[dr, dashed, "\cong"] \arrow[drr, bend left] \arrow[ddr, bend right, "p"'] & & \\ & C.\chi_p \arrow[r, "q_{\chi_p}"] \arrow[d, "p_{\chi_p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{V} \arrow[d, "\pi"] \\ & C \arrow[r, "\chi_p"'] & V. \end{tikzcd} \tag{2.3}$$

In these circumstances, we say that the stable map $p : D \twoheadrightarrow C$ is *displayed* (by $p_{\chi_p} : C.\chi_p \twoheadrightarrow C$).

We note that the term *display map* originates in [Tay87], where it refers to members of a stable class of maps satisfying some futher conditions. Our terminology follows [Awo16] and [AGH24].

2.1.10. *Notation.* We write D to denote the class of display maps for a classifier $\pi : \widetilde{V} \twoheadrightarrow V$ of stable maps S. We note that the closure of D under isomorphism is the class S (recalling Observation 2.1.6).

2.1.11. EXAMPLE (Hofmann-Streicher universe). Continuing Example 2.1.5, the presheaf category $\widehat{\mathbb{C}}$ admits a classifier for small maps, namely the *Hofmann-Streicher universe* [HS97]. Let c be an object in $\mathbb{C}$ and define presheaves U and E by

$$
\begin{aligned}
U(c) &:= \mathsf{Obj}[(\mathbb{C}/c)^{\mathsf{op}}, \mathsf{Set}_\kappa] \\
E(c) &:= \mathsf{Obj}[(\mathbb{C}/c)^{\mathsf{op}}, \mathsf{Set}_\kappa^\bullet]
\end{aligned}
$$

where Set_κ is the category of κ-small sets and functions, and $\mathsf{Set}_\kappa^\bullet$ is the pointed version thereof (Terminology 2.1.1). The action of an arrow $f : d \to c$ in $\mathcal{C}$ is by precomposition with the composition functor from $\mathcal{C}/d \to \mathcal{C}/c$. A natural transformation

$$\pi : E \to U$$

is induced by composition with the forgetful functor $\mathsf{Set}_\kappa^\bullet \to \mathsf{Set}_\kappa$.

2.1.12. *Remark.* The Hofmann-Streicher universe is a classifer for the class of small maps $\mathcal{S}$ from Example 2.1.5. Following [AGH24, Section 1], this can be viewed as structure on the map $\pi : E \to U$, in the sense of Remark 2.1.8. For any small map $p : A \twoheadrightarrow X$, there is a canonical map $\alpha : X \to U$ that is the classifying map of p. This is defined, for an object c in $\mathbb{C}$ and $x \in X(c)$, by an element $\alpha_x : \mathbb{C}/c^{\mathsf{op}} \to \mathsf{Set}_\kappa$ in $U(c)$, itself given at $f : d \to c$ by

$$\alpha_x(f) := \{a \in A(d) \mid p_d(a) = X(f)(x)\}.$$

There are also canonical pullback squares of $\pi : E \to U$ along each $\alpha : X \to U$. When X is a representable presheaf, that is, $\alpha : よc \to U$, there is the pullback square

$$
\begin{tikzcd}
よc.\alpha \arrow[r] \arrow[d, two heads, "p_\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "\pi"] \\
よc \arrow[r, "\alpha"'] & U
\end{tikzcd}
\tag{2.4}
$$

in which $よc.\alpha$ at d in $\mathbb{C}$ is taken to be $よc.\alpha(d) := \coprod_{f \in \mathsf{Hom}_{\mathbb{C}}(d,c)} \alpha(f)$. This determines a choice of pullback square for each $\alpha : X \to U$, denoted

$$
\begin{tikzcd}
X.\alpha \arrow[r] \arrow[d, two heads, "p_\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "\pi"] \\
X \arrow[r, "\alpha"'] & U.
\end{tikzcd}
\tag{2.5}
$$

Recalling Terminology 2.1.9, the object $X.\alpha$ is the comprehension of α with respect to U and the map $p_\alpha : X.\alpha \twoheadrightarrow X$ is the display map associated to α. It follows that every small map is isomorphic to a display map, so for every small map $p : A \twoheadrightarrow X$, there is a diagram

$$\begin{tikzcd} A \arrow[dr, dashed, "\cong"] \arrow[drr, bend left] \arrow[ddr, bend right, "p"'] & & \\ & X.\alpha \arrow[r, "q_\alpha"] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, "\pi"] \\ & X \arrow[r, "\alpha"'] & U, \end{tikzcd} \tag{2.6}$$

in which $p : A \to X$ is displayed by $p_\alpha : X.\alpha \to X$. In summary, there are three equivalent notions:

(i) a stable map $p : A \twoheadrightarrow X$

(ii) a map $\alpha : X \to U$

(iii) a display map $p_\alpha : X.\alpha \twoheadrightarrow X$.

From Chapter 4 onwards, we will routinely move between these points of view.

Natural model

Let $\mathcal{C}$ be a small category. The main gadget of a natural model lives in the category $\widehat{\mathcal{C}}$ of presheaves over $\mathcal{C}$.

2.1.13. DEFINITION (Representable natural transformation). A morphism $f : Y \to X$ in $\widehat{\mathcal{C}}$ is a *representable natural transformation* if all of its fibres are representable objects; that is, for every object C in $\mathcal{C}$ and every element $x \in X(C)$, there is an object D in $\mathcal{C}$, a map $p : D \to C$ in $\mathcal{C}$, and an element $q \in Y(D)$ such that the following square is a pullback in $\widehat{\mathcal{C}}$:

$$\begin{tikzcd} D \arrow[r, "y"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner" very near start] & Y \arrow[d, "f"] \\ C \arrow[r, "x"'] & X. \end{tikzcd} \tag{2.7}$$

2.1.14. *Remark.* Just like Convention 2.1.8, in which a classifier of a class of stable maps could be viewed as a property or as structure, the notion of representability of a natural transformation $f : Y \to X$ may be viewed in either of these ways. In the structured case, we suppose that there is a specified function that, to each object $X \in \mathcal{C}$ and each element $\sigma \in U(X)$, assigns an object $X.\sigma$, a map p_σ, and element $q_\sigma \in \widetilde{\mathcal{U}}(X.\sigma)$ forming a pullback square

$$\begin{tikzcd} X.\sigma \arrow[r, "q_\sigma"] \arrow[d, "p_\sigma"'] \arrow[dr, phantom, "\lrcorner" very near start] & \widetilde{\mathcal{U}} \arrow[d, "\mathsf{ty}"] \\ X \arrow[r, "\sigma"'] & \mathcal{U}. \end{tikzcd} \tag{2.8}$$

To describe the structured version of a representable natural transformation, we introduce the following terminology.

2.1.15. *Terminology.* (i) We refer to a pullback square as in (2.7) as a *represented pullback*. A map $p : D \to C$ in $\mathcal{C}$ for which there exists a $y \in Y(D)$ fitting into such a pullback square will be called a *representative* for f along $x \in X(C)$.

(ii) We refer to a pullback square as in (2.8) as the *specified represented pullback* for the object X in $\mathcal{C}$ and the element $\sigma \in \mathcal{U}(X)$, and such a map $p_\sigma : X.\sigma \to X$ will be called the *specified* *representative* or *display map* for σ. Using the term *display map* is justified by the fact that it plays the same role as a display map in the context of a category with a classified stable class of maps (Terminology 2.1.9).

2.1.16. *Notation.* For a representable natural transformation $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $\widehat{\mathcal{C}}$, we denote the associated class of representatives by R. For the structured version, we denote the class of specified representatives by D.

We now define a natural model of dependent type theory.

2.1.17. DEFINITION (Natural model [Fio12; Awo12; Awo16]). Let $\mathcal{C}$ be a category with a terminal object. A *natural model of type theory* on $\mathcal{C}$ is a representable natural transformation $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $\widehat{\mathcal{C}}$.

2.1.18. *Convention.* It is standard in modelling the core features of dependent type theory to require that the category has a terminal object, capturing the empty context. The structured version of this, in the sense of Remark 2.1.14, is a specified terminal object.

2.1.19. *Remark.* Calling a representable natural transformation a *natural model of type theory* is justified in [Awo16] by the observation that it is the same as a *category with families* (cwf) [Dyb95]. A cwf is a variable-free presentation of the core of dependent type theory: namely, (i) contexts and substitutions, (ii) types and terms in context, and (iii) context extension. The correspondence between a cwf and a natural model on $\mathcal{C}$, given by $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $\widehat{\mathcal{C}}$, is as follows.

(i) Objects and arrows of $\mathcal{C}$ correspond to contexts and substitutions respectively, made most clear by an explicit substitution presentation of type theory,[2] as in the notation:

$$\begin{array}{rcl} \Gamma \vdash & \leftrightsquigarrow & \Gamma \in \mathsf{obj}(\mathcal{C}) \\ \Delta \vdash s : \Gamma & \leftrightsquigarrow & s : \Delta \to \Gamma \in \mathsf{mor}(\mathcal{C}). \end{array}$$

In particular, the terminal object corresponds to the empty context:

$$\cdot \vdash \quad \leftrightsquigarrow \quad 1 \in \mathsf{obj}(\mathcal{C})$$

(ii) The elements of the representable natural transformation are written

$$\begin{array}{rcl} \Gamma \vdash \alpha \ \mathsf{type} & \leftrightsquigarrow & \alpha \in \mathcal{U}(\Gamma) \\ \Gamma \vdash a : \alpha & \leftrightsquigarrow & a \in \widetilde{\mathcal{U}}(\Gamma) \text{ and } \alpha = \mathsf{ty} \circ a, \end{array}$$

or diagrammatically:

$$\begin{tikzcd} & \widetilde{\mathcal{U}} \arrow[d, "\mathsf{ty}"] \\ \Gamma \arrow[ur, "a"] \arrow[r, "\alpha"'] & \mathcal{U}. \end{tikzcd}$$

In this way, $\mathcal{U}(\Gamma)$ is the set of types in context Γ; $\widetilde{\mathcal{U}}(\Gamma)$ is the set of typed terms in context Γ, that is, pairs (a, α) with $a : \alpha$; and the component of the representable natural transformation $\mathsf{ty}_\Gamma : \widetilde{\mathcal{U}}(\Gamma) \to \mathcal{U}(\Gamma)$ is projection of the type of a typed term in context Γ. The action of a

[2] See, for example, [CGH14]

map $s : \Delta \to \Gamma$ on an element $\alpha \in \mathcal{U}(\Gamma)$ or $a \in \widetilde{\mathcal{U}}(\Gamma)$ corresponds to substitution, denoted $\alpha[s] \in \mathcal{U}(\Delta)$ and $a[s] \in \widetilde{\mathcal{U}}(\Delta)$ respectively. The naturality of ty ensures that the term substitution rule is validated, that is, the commutativity of the following diagrams validate corresponding rules:

$$\frac{\Gamma \vdash \alpha \text{ type} \qquad s : \Delta \to \Gamma}{\Delta \vdash \alpha[s] \text{ type}} \quad \leftrightsquigarrow \quad \Delta \xrightarrow{s} \Gamma \xrightarrow{\alpha} \mathcal{U}$$

$$\frac{\Gamma \vdash a : \alpha \qquad s : \Delta \to \Gamma}{\Delta \vdash a[s] : \alpha[s]} \quad \leftrightsquigarrow \quad \begin{tikzcd} & & \widetilde{\mathcal{U}} \arrow[d, "\mathrm{ty}"] \\ \Delta \arrow[r, "s"'] & \Gamma \arrow[ur, "a"] \arrow[r, "\alpha"'] & \mathcal{U}. \end{tikzcd}$$

Given further a map $t : \Delta' \to \Delta$, functoriality means the associativity of substitution is validated.

(iii) Finally, context extension is the representability of ty:

$$\begin{tikzcd} \Gamma.\alpha \arrow[r, "v_\alpha"] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "\mathrm{ty}"] \\ \Gamma \arrow[r, "\alpha"'] & \mathcal{U} \end{tikzcd} \quad \leftrightsquigarrow \quad \frac{\Gamma \vdash \alpha \text{ type}}{\Gamma.\alpha \vdash v_\alpha : \alpha[p_\alpha]}$$

where v_α is a variable of type α, but in the extended context. The universal property of the above pullback square is precisely the universal property of context extension in [Dyb95].

2.1.20. EXAMPLE ([Awo16, Section 3]). Let $\mathcal{C}$ be a category with a terminal object and a stable class of maps S. Then there is a natural model of type theory on $\mathcal{C}$ with class of represented pullbacks S, given by a representable natural transformation constructed as follows. Let $\mathsf{S}_0, \mathsf{S}_1$ be presheaves defined at an object C in $\mathcal{C}$ by

$$\mathsf{S}_1(C) := \{(a, d) \in \mathsf{mor}(\mathcal{C}) \times \mathsf{S} \mid \mathsf{cod}(a) = \mathsf{dom}(d)\}$$
$$\mathsf{S}_0(C) := \{(b, d) \in \mathsf{mor}(\mathcal{C}) \times \mathsf{S} \mid \mathsf{cod}(b) = \mathsf{cod}(d)\},$$

or diagrammatically as follows:

$$\mathsf{S}_1(C) := \left\{ \begin{tikzcd} & D' \arrow[d, "d"] \\ C \arrow[ur, "a"] & D \end{tikzcd} \right\} \qquad \mathsf{S}_0(C) := \left\{ \begin{tikzcd} & D' \arrow[d, "d"] \\ C \arrow[r, "b"'] & D \end{tikzcd} \right\}$$

There is a representable natural transformation $\pi : \mathsf{S}_1 \to \mathsf{S}_0$ with component at C in $\mathcal{C}$

$$\begin{aligned} \mathsf{S}_1(C) &\xrightarrow{\pi_C} \mathsf{S}_0(C) \\ (a, d) &\longmapsto (d \circ a, d). \end{aligned}$$

The action of presheaves is by precomposition in the first factor, that is, for $s : C' \to C$,

$$\mathsf{S}_1(s)(a, d) = (a \circ s, d) \quad \text{and} \quad \mathsf{S}_0(s)(b, d) = (b \circ s, d),$$

which is strictly functorial.

For a proof that $\pi : \mathsf{S}_1 \to \mathsf{S}_0$ is a representable natural transformation with class of representatives S, see [Awo16, Proposition 23].

Relationship between categories with classified stable maps and natural models

We have just seen that a category with a class of stable maps S gives rise to a representable natural transformation with class of representatives S (Example 2.1.20). In the other direction, the class of representatives associated with a representable natural transformation is a stable class of maps. This is the content of the next proposition, the substance of which is already contained in [Awo16, Section 3] but is reexpressed here for the purposes of our exposition.

2.1.21. PROPOSITION. *Let $\alpha : Y \to X$ be a representable natural transformation in $\widehat{\mathcal{C}}$ and let R be the associated class of representatives, as in Terminology 2.1.15. Then R is a stable class of maps in $\mathcal{C}$.*

Proof. Let $f : C' \to C$ be an arbitrary map in $\mathcal{C}$ and let $p : D \twoheadrightarrow C$ be a member of R. Since $\alpha : Y \to X$ is a representable natural transformation, there exists a represented pullback along $\text{よ} f \circ x$ in $\widehat{\mathcal{C}}$:

$$\begin{tikzcd} \text{よ}D' \arrow[r, dashed] \arrow[d, "\text{よ}p'"'] \arrow[rr, bend left, "y'"] & \text{よ}D \arrow[r, "y"] \arrow[d, "\text{よ}p"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\alpha"] \\ \text{よ}C' \arrow[r, "\text{よ}f"'] & \text{よ}C \arrow[r, "x"'] & X, \end{tikzcd} \tag{2.9}$$

The universal property of the right-hand pullback induces the unique dashed arrow $\text{よ}D' \to \text{よ}D$ which, by the pasting law for pullbacks, makes the left-hand square into a pullback. Since the Yoneda embedding preserves and reflects limits, the left-hand square gives a pullback square in $\mathcal{C}$. □

It remains to describe the relationship between the notion of type universe in each of the models, one living in the category $\mathcal{C}$ itself and the other in the presheaf category $\widehat{\mathcal{C}}$. The next proposition says that a type universe over a category $\mathcal{C}$, in the sense of a representable natural transformation, gives rise to a type universe in the category $\mathcal{C}$, in the sense of a classifier, if the representable natural transformation is literally representable. To the best of our knowledge, this result does not already appear in the literature.

2.1.22. PROPOSITION. *Let $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ be a representable natural transformation in $\widehat{\mathcal{C}}$ and let R be the associated class of representatives, as in Terminology 2.1.15. Then $\mathcal{C}$ admits a classifier for R if the presheaves $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ are representable.*

Proof. Suppose that $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ are represented respectively by objects $\widetilde{W}$ and W in $\mathcal{C}$, with natural isomorphisms given by

$$\mathcal{U} \overset{\alpha}{\cong} \mathsf{Hom}_{\mathcal{C}}(-, W) \quad \text{and} \quad \widetilde{\mathcal{U}} \overset{\beta}{\cong} \mathsf{Hom}_{\mathcal{C}}(-, \widetilde{W}).$$

From the isomorphism

$$\mathsf{Hom}_{\mathcal{C}}(\widetilde{W}, W) \cong \mathsf{Hom}_{\widehat{\mathcal{C}}}(\text{よ}\widetilde{W}, \text{よ}W) \tag{2.10}$$

induced by the Yoneda embedding, we obtain a map $\pi : \widetilde{W} \to W$ in $\mathcal{C}$ by taking the composition

$$\mathsf{Hom}_{\mathcal{C}}(-, \widetilde{W}) \xrightarrow[\cong]{\beta^{-1}} \widetilde{\mathcal{U}} \xrightarrow{\mathsf{ty}} \mathcal{U} \xrightarrow[\cong]{\alpha} \mathsf{Hom}_{\mathcal{C}}(-, W)$$

in $\widehat{\mathcal{C}}$ and evaluating its component at $\widetilde{W}$ at $\mathsf{id}_{\widetilde{W}}$:

$$\pi := (\alpha \circ \mathsf{ty} \circ \beta^{-1})_{\widetilde{W}}(\mathsf{id}_{\widetilde{W}}).$$

To see that π is a classifier for R—which is a stable class of maps by Proposition 2.1.21—we first show that it belongs to R. Consider the following diagram in $\widehat{\mathcal{C}}$:

$$\begin{tikzcd} \text{よ}\widetilde{W} \arrow[r, "\beta^{-1}"] \arrow[d, "\text{よ}\pi"'] & \widetilde{\mathcal{U}} \arrow[d, "\mathsf{ty}"] \\ \text{よ}W \arrow[r, "\alpha^{-1}"'] & \mathcal{U}. \end{tikzcd} \tag{2.11}$$

This commutes, since $\text{よ}\pi = \alpha \circ \mathsf{ty} \circ \beta^{-1}$ by the isomorphism of Equation 2.10. Furthermore, it is a pullback as a consequence of α and β being isomorphisms. Therefore, π appears in a represented pullback and so belongs to R.

To show that π is a classifier, let $p : Y \rightarrowtriangle X$ be a member of R. Then there are elements $u \in \mathcal{U}(X)$ and $q \in \widetilde{\mathcal{U}}$ forming the following pullback square in $\widehat{\mathcal{C}}$:

$$\begin{tikzcd} \text{よ}Y \arrow[r, "q"] \arrow[d, "\text{よ}p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \widetilde{\mathcal{U}} \arrow[d, "\mathsf{ty}"] \\ \text{よ}X \arrow[r, "u"'] & \mathcal{U}. \end{tikzcd} \tag{2.12}$$

Using the isomorphisms α and β, we obtain the following commutative diagram:

$$\begin{tikzcd} \text{よ}Y \arrow[r, "\beta \circ q"] \arrow[d, "\text{よ}p"'] & \text{よ}\widetilde{W} \arrow[r, "\beta^{-1}"] \arrow[d, "\text{よ}\pi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \widetilde{\mathcal{U}} \arrow[d, "\mathsf{ty}"] \\ \text{よ}X \arrow[r, "\alpha \circ u"'] & \text{よ}W \arrow[r, "\alpha^{-1}"'] & \mathcal{U}. \end{tikzcd} \tag{2.13}$$

By the pasting law for pullbacks, the left-hand square is also a pullback. Since the Yoneda embedding is full and faithful, the left-hand square is mapped to by a commutative diagram in $\mathcal{C}$,

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p"', -open triangle] & \widetilde{W} \arrow[d, "\pi", -open triangle] \\ X \arrow[r] & W, \end{tikzcd}$$

which is a pullback because よ reflects limits. Therefore, π is a classifier for the stable class of maps R. ☐

We note the following corollary, to be used in Section 3.3.

2.1.23. COROLLARY. *The Yoneda embedding of a classifier is a representable natural transformation with class of representatives equal to the original class of stable maps.*

Proof. This follows from Proposition 2.1.22 by replacing the natural isomorphisms α and β with strict equalities. ☐

2.2 DUAL-CONTEXT TYPE THEORY AS AN INTERNAL LANGUAGE

Recall that we wish to identify categories for which an internal language can be given by crisp type theory, the fragment of Shulman's spatial type theory [Shu18] singled-out in [LOPS18] for describing internal universes of fibrant types. Crisp type theory is a *dual-context type theory*, meaning contexts are dependent lists of variable-type pairs separated into two context zones,

$$\Delta \mid \Gamma = x_1 :: \delta_1, \dots, x_n :: \delta_n \mid y_1 : \gamma_1, \dots, y_m : \gamma_m,$$

with rules treating variables in each zone differently. Given this context structure, the models of dependent type theory in the previous section do not immediately apply. In this section, we develop an appropriate version of model for dual-context type theory in order to understand how it can act as an internal language. This involves equipping two categories with the structure to model dependent type theory—one modelling the first context zone and the other modelling the entire context—and then specifying the appropriate relationship between these type theories to capture dual-context type theory.

In order to develop this material, we begin by recalling some concepts from fibred category theory. We then proceed in a modular way, looking at different facets of dual-context type theory: the dependency of the second context zone on the first; empty contexts; extension of the second context zone; extension of the first context zone; and weakening in the first context zone. These correspond to the rules in Figure 1.3. For each feature, we follow the same pattern: we begin with a syntactic rule and isolate a categorical counterpart, then we remark on how this categorical structure provides an interpretation of the rule and finally how it can be used to generate an internal language. The last step—extracting an internal language—is a process we will do in detail in Chapter 4. The development in this section will then be summarised in a definition of model in Section 2.3.

Fibred categories

Let $\mathcal{E}$ and $\mathcal{B}$ be small categories. We introduce some notions from fibred category theory to be used to model dual-context type theory, using [Jac99] as a reference.

2.2.1. *Remark.* A functor $P : \mathcal{E} \to \mathcal{B}$ may be viewed as a *family of categories*, in the sense that for each object B in $\mathcal{B}$, there exists a category consisting of objects E in $\mathcal{E}$ with $P(E) = B$ and arrows $f : E' \to E$ in $\mathcal{E}$ with $P(f) = \mathsf{id}_B$.

When viewing a functor as a family of categories, we adopt the following terminology.

2.2.2. *Terminology.* Let $P : \mathcal{E} \to \mathcal{B}$ be a functor.

(i) For an object B in $\mathcal{B}$, we say that the category consisting of objects E in $\mathcal{E}$ with $P(E) = B$ and arrows $f : E' \to E$ in $\mathcal{E}$ with $P(f) = \mathsf{id}_B$ is the *fibre (category) over* B, denoted $\mathcal{E}_B$.

(ii) Objects and morphisms in $\mathcal{E}_B$ are said to *live above* B.

(iii) A map $f : E' \to E$ in $\mathcal{E}$ is called a *vertical morphism* if it lives in a fibre category, that is, above some identity morphism in $\mathcal{B}$.

(iv) We call $\mathcal{E}$ the *total category* and $\mathcal{B}$ the *base category*.

2.2.3. DEFINITION (cartesian morphism). Let $P : \mathcal{E} \to \mathcal{B}$. An arrow $f : E' \to E$ in $\mathcal{E}$ is *cartesian* if for any $g : E'' \to E$ in $\mathcal{E}$ and $w : P(E'') \to P(E)$ in $\mathcal{B}$ with $P(f) \circ w = P(g)$, there exists a unique $\widehat{w} : E'' \to E'$ lying over w and with $f \circ \widehat{w} = g$.

$$\begin{tikzcd}
 & E'' \arrow[dr, dashed, "\widehat{w}"'] \arrow[drr, bend left, "g"] & & \\
\mathcal{E} \arrow[dd, "P"'] & & E' \arrow[r, "f"'] & E \\
 & P(E'') \arrow[dr, "w"'] \arrow[drr, bend left, "P(f)\circ w = P(g)"] & & \\
\mathcal{B} & & P(E') \arrow[r, "P(f)"'] & P(E)
\end{tikzcd} \tag{2.14}$$

2.2.4. *Terminology.* Let $u : B' \to B$ be a map in $\mathcal{B}$ and E be an object in the fibre over B. We say that u has a *cartesian lift* to E with respect to P if there is a cartesian morphism f in $\mathcal{E}$ with $P(f) = u$.

After considering what is required of a functor $P : \mathcal{E} \to \mathcal{B}$ to model each feature of the dual-context structure, we will be interested in the extent to which P is a *Grothendieck fibration* (Remark 2.3.5), a device commonly used in modelling dependent type theory.

2.2.5. DEFINITION (Grothendieck fibration). A functor $P : \mathcal{E} \to \mathcal{B}$ is a *Grothendieck fibration* if for all objects E in $\mathcal{E}$ and arrows $f : B \to P(E)$ in $\mathcal{B}$, f has a cartesian lift.

We now establish how a fibred perspective can be used to model dual-context type theory, beginning with the dependency of the second context zone on the first.

Dependency of the zones in a dual-context

The idea is to capture the dependency of the second context zone on the first via a functor $P : \mathcal{E} \to \mathcal{B}$ viewed as a family of categories, in the sense of Remark 2.2.1. Each fibre in $\mathcal{E}$ will correspond to a category of contexts (as in Remark 2.1.4) with a *fixed* first context zone. In other words, an object B in $\mathcal{B}$ will correspond to the first part Δ of a dual context $\Delta \mid \Gamma$, while objects in the fibre $\mathcal{E}_B$ will correspond to the entire context. In this way, each of the fibres and the base category should model standard dependent type theories, but with attention paid to how these type theories relate to each other.

2.2.6. *Remark.* Given a functor $P : \mathcal{E} \to \mathcal{B}$, we have the following intended interpretation of contexts in the standard dependent type theory associated with the first context zone, and dual-contexts:

$$\begin{aligned} \Delta \vdash \quad &\leftrightsquigarrow \quad B \in \mathsf{obj}(\mathcal{B}) \\ \Delta \mid \Gamma \vdash \quad &\leftrightsquigarrow \quad E \in \mathsf{obj}(\mathcal{E}). \end{aligned}$$

For internal languages associated with $\mathcal{B}$ and $\mathcal{E}$, we introduce the following notation:

$$\begin{aligned} B \vdash \quad &\leftrightsquigarrow \quad B \in \mathsf{obj}(\mathcal{B}) \\ E \vdash_B \quad &\leftrightsquigarrow \quad E \in \mathsf{obj}(\mathcal{E}) \text{ and } P(E) = B. \end{aligned}$$

Empty contexts

Recall that in standard type theory, an empty context is modelled by a terminal object. Under the view that objects in the fibre $\mathcal{E}_B$ for a functor $P : \mathcal{E} \to \mathcal{B}$ correspond to contexts with the same first context zone, to model a context with no variables in the second zone, as in the rule

$$\textsc{ctx/emp}\quad \frac{\Delta\ \mathsf{ctx}}{\Delta \mid \cdot\ \mathsf{ctx}}$$

corresponds to having a terminal object in the fibre. The fibres must be equipped with terminal objects in a coherent way, motivating the following definition.

2.2.7. DEFINITION (stable terminal objects in fibres). Let $P : \mathcal{E} \to \mathcal{B}$ be a functor. The fibres of P have *stable terminal objects* if P has a right adjoint right inverse $T : \mathcal{B} \to \mathcal{E}$.

2.2.8. *Observation.* A functor $P : \mathcal{E} \to \mathcal{B}$ having a right adjoint right inverse equivalently says that for all objects E in $\mathcal{E}$ and maps $f : P(E) \to B$ in $\mathcal{B}$, there exists a unique map $\hat{f} : E \to T(B)$ with $P(\hat{f}) = f$.

This observation justifies the terminology of Definition 2.2.7: each fibre $\mathcal{E}_B$ has a terminal object $T(B)$, since for every object in the fibre there is a unique map $E \to T(B)$. Furthermore, fibrewise terminal objects admit reindexing along all arrows $f : B' \to B$, and terminal objects are stable under this reindexing. This is the content of the next proposition.

2.2.9. PROPOSITION. *Let $f : B' \to B$ in $\mathcal{B}$. Then $T(f) : T(B') \to T(B)$ is a cartesian lift of f.*

Proof. Firstly, $T(f)$ lies over f since T is right inverse to P. Suppose there is a map $g : E \to T(B)$ and a commutative triangle

$$\begin{tikzcd}
P(E) \arrow[dr, "h"'] \arrow[drr, bend left, "P(g)"] & & \\
& B' \arrow[r, "f"'] & B.
\end{tikzcd}$$

Then by Observation 2.2.8, the universal property of the right adjoint right inverse means there exists a unique map $\hat{h} : E \to T(B')$, which forms the following commutative diagram in $\mathcal{E}$

$$\begin{tikzcd}
E \arrow[dr, "\hat{h}"'] \arrow[drr, bend left, "g"] & & \\
& T(B') \arrow[r, "T(f)"'] & T(B)
\end{tikzcd}$$

because $\hat{h}$ lies over h, so

$$P(T(f)) \circ h = P(T(f)) \circ P(\hat{h}) = f \circ h = P(g).$$

$\square$

Finally, to model the context with both context zones empty,

$$\cdot \mid \cdot$$

requires a terminal object $1_{\mathcal{B}}$ in the base category $\mathcal{B}$. This is then promoted to a dual-context in the total category by applying the functor $T : \mathcal{B} \to \mathcal{E}$. It is straightforward to see that the object $T(1_{\mathcal{B}})$ in $\mathcal{E}$ is a terminal object in both the fibre category $\mathcal{E}_{1_{\mathcal{B}}}$ and the total category $\mathcal{E}$.

2.2.10. *Remark.* The categorical structure of Definition 2.2.7 provides the following interpretation of context judgements on the left as objects in $\mathcal{E}$ and $\mathcal{B}$ on the right:

$$\begin{aligned} \cdot \vdash \quad &\leftrightsquigarrow \quad 1_{\mathcal{B}} \in \mathsf{obj}(\mathcal{B}) \\ \Delta \mid \cdot \vdash \quad &\leftrightsquigarrow \quad T(B) \in \mathcal{E} \text{ for some } B \in \mathsf{obj}(\mathcal{B}) \\ \cdot \mid \cdot \vdash \quad &\leftrightsquigarrow \quad T(1_{\mathcal{B}}) \in \mathcal{E}. \end{aligned}$$

Continuing the internal language notation from Remark 2.2.6, we have the following context judgements in the internal languages associated with $P : \mathcal{E} \to \mathcal{B}$:

$$\begin{aligned} 1_{\mathcal{B}} \vdash \quad &\leftrightsquigarrow \quad 1_{\mathcal{B}} \in \mathsf{obj}(\mathcal{B}) \\ T(B) \vdash_B \quad &\leftrightsquigarrow \quad T(B) \in \mathcal{E} \text{ for some } B \in \mathsf{obj}(\mathcal{B}) \\ T(1_{\mathcal{B}}) \vdash_{1_{\mathcal{B}}} \quad &\leftrightsquigarrow \quad T(1_{\mathcal{B}}) \in \mathcal{E}. \end{aligned}$$

Context extension in the second context zone

Extension of the second context zone in a dual-context type theory, as in the rule

$$\textsc{ctx/ext/2nd}\quad \frac{\Delta \mid \Gamma \vdash \sigma \ \mathsf{type}}{\Delta \mid \Gamma, x : \sigma \ \mathsf{ctx}}$$

is like standard context extension in that it simply appends a variable to the end of a list. Since the first part of the context is unchanged, the object corresponding to the extended context should be in the same fibre as the original context. A category $\mathcal{C}$ models standard context extension if there is a representable natural transformation in $\widehat{\mathcal{C}}$. Thus we will model extension of the second context zone by equipping each fibre with a representable natural transformation, in a coherent way, as follows.

2.2.11. DEFINITION (fibrewise representable natural transformation). Let $P : \mathcal{E} \to \mathcal{B}$ be a functor and $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ be a representable natural transformation in $\widehat{\mathcal{E}}$. Then ty is a *fibrewise representable natural transformation* with respect to P if all of the representatives associated with ty (Terminology 2.1.15(i)) are vertical maps. That is, for every object E in $\mathcal{E}$ and every element $\sigma \in \mathcal{U}(E)$, the specified choice of map $p_\sigma : E.\sigma \to E$ in $\mathcal{E}$ fitting into a pullback square

$$\begin{tikzcd} E.\sigma \arrow[r, "v_\sigma"] \arrow[d, "p_\sigma"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \widetilde{\mathcal{U}} \arrow[d, "\mathsf{ty}"] \\ E \arrow[r, "\sigma"'] & \mathcal{U}, \end{tikzcd} \tag{2.15}$$

is in the fibre $\mathcal{E}_{P(E)}$, that is,

$$P(p_\sigma) = \mathsf{id}_{P(E)}.$$

2.2.12. *Terminology.* Extending Terminology 2.1.15, a vertical map in $\mathcal{E}$ is said to be a *vertical display map* (resp. *vertical representative map*) if it is a display map (resp. *representative map*) associated with ty.

In the next definition, we combine fibrewise representable natural transformation with the stable terminal objects from Definition 2.2.7 into a notion of *fibrewise natural model*.

2.2.13. DEFINITION. Let $P : \mathcal{E} \to \mathcal{B}$ be a functor. A *fibrewise natural model structure* on $\mathcal{E}$ with respect to P consists of a right adjoint right inverse $T : \mathcal{B} \to \mathcal{E}$ and a P-fibrewise representable natural transformation $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $\widehat{\mathcal{E}}$.

We justify the terminology in this definition with the following proposition.

2.2.14. PROPOSITION. *Let $P : \mathcal{E} \to \mathcal{B}$ be a functor and $(T : \mathcal{B} \to \mathcal{E}, \mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U})$ be the data of a fibrewise natural model strucutre on $\mathcal{E}$. Then for each B in $\mathcal{B}$, there is a natural model on the fibre $\mathcal{E}_B$. Furthermore, this structure is preserved between the fibres.*

Proof. There is a natural model on each fibre $\mathcal{E}_B$, given by the specifed terminal object $T(B)$ and the restriction of ty to the fibre,

$$\mathsf{ty}|_{\mathcal{E}_B} : \widetilde{\mathcal{U}}|_{\mathcal{E}_B} \to \mathcal{U}|_{\mathcal{E}_B}.$$

Specified terminal objects are preserved by substitution by Proposition 2.2.9. Specified represented pullbacks are preserved in the following sense. Let $f : E' \to E$ be any map in $\mathcal{E}$ with E' in the fibre $\mathcal{E}_{B'}$, where B' may be distinct from B. Let $(\sigma \in \mathcal{U}(E), p_\sigma : E.\sigma \to E, q_\sigma \in \widetilde{\mathcal{U}}(E.\sigma))$ be the data of a specified represented pullback in $\widehat{\mathcal{E}}_B$. Then there is a pullback in $\mathcal{E}$

$$\begin{tikzcd}
E.\sigma[f] \arrow[r] \arrow[d, "p_{\sigma[f]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E.\sigma \arrow[d, "p_\sigma"] \\
E' \arrow[r, "f"'] & E
\end{tikzcd}$$

since $p_\sigma[f]$ is part of a specified represented pullback. □

2.2.15. *Remark.* Note that *fibrewise natural model structure* is an intermediate notion for the purposes of the modular development in this section. It will be superseded in the next section by the definition (2.3.1) of *fibred natural model of dual-context type theory*.

2.2.16. *Remark.* Recalling Remark 2.1.19, in the situation of the pullback square in (2.15), we have the following intended interpretation of dual-context judgements:

$$\begin{aligned}
\Delta \mid \Gamma \vdash \sigma \ \mathsf{type} \quad &\leftrightsquigarrow \quad \sigma \in \mathcal{U}(E) \\
\Delta \mid \Gamma \vdash t : \sigma \quad &\leftrightsquigarrow \quad t \in \widetilde{\mathcal{U}}(E) \text{ and } \sigma = \mathsf{ty} \circ t \\
\Delta \mid \Gamma.\sigma \vdash v_\sigma : \sigma[p_\sigma] \quad &\leftrightsquigarrow \quad \mathsf{ty} \circ v_\sigma = \sigma \circ p_\sigma
\end{aligned}$$

We also continue the internal language notation as follows:

$$\begin{aligned}
E \vdash_B \sigma \ \mathsf{type} \quad &\leftrightsquigarrow \quad \sigma \in \mathcal{U}(E) \text{ and } P(E) = B \\
E \vdash_B t : \sigma \quad &\leftrightsquigarrow \quad t \in \widetilde{\mathcal{U}}(E), \sigma = \mathsf{ty} \circ t \text{ and } P(E) = B \\
E.\sigma \vdash_B v_\sigma : \sigma[p_\sigma] \quad &\leftrightsquigarrow \quad \mathsf{ty} \circ v_\sigma = \sigma \circ p_\sigma \text{ and } P(E) = B
\end{aligned}$$

Context extension in the first context zone

When the second part of the context is empty, the first context zone may be extended, as in the following rule:

$$\frac{\Delta \mid \cdot \vdash \sigma \ \mathsf{type}}{\Delta, x :: \sigma \mid \cdot \ \mathsf{ctx}} \ \textsc{ctx/ext/1st}$$

Combining Remark 2.2.10 on empty contexts and Remark 2.2.16 on type judgements, the intended interpretation of the premise of the rule is an element $\sigma \in \mathcal{U}(T(B))$, for some object B in $\mathcal{B}$. This should be the same as the interpretation of the judgement

$$\Delta \vdash \sigma \ \mathsf{type}$$

in the standard dependent type theory associated with the first context zone. Therefore, the base category $\mathcal{B}$ should be endowed with a natural model structure *relative* to that on the total category $\mathcal{E}$, for which we make the following observation.

2.2.17. *Observation.* By the Yoneda lemma, an element $\sigma \in \mathcal{U}(T(B))$ is isomorphic to a map of presheaves

$$T(B) \xrightarrow{\sigma} \mathcal{U}$$

in $\widehat{\mathcal{E}}$, but also to a map

$$B \xrightarrow{\sigma} \mathcal{U} \circ T^{\mathsf{op}} \tag{2.16}$$

in $\widehat{\mathcal{B}}$. Therefore, there is a natural isomorphism of hom-sets

$$\mathsf{Hom}_{\widehat{\mathcal{E}}}(T(B), \mathcal{U}) \cong \mathsf{Hom}_{\widehat{\mathcal{B}}}(B, \mathcal{U} \circ T^{\mathsf{op}}).$$

Note that in (2.16), we take the opposite functor $T^{\mathsf{op}} : \mathcal{B}^{\mathsf{op}} \to \mathcal{E}^{\mathsf{op}}$ to get a map of presheaves over $\mathcal{B}$,

$$\mathsf{ty} \circ T^{\mathsf{op}} : \widetilde{\mathcal{U}} \circ T^{\mathsf{op}} \to \mathcal{U} \circ T^{\mathsf{op}},$$

given diagrammatically as

$$\begin{tikzcd} \mathcal{B}^{\mathsf{op}} \arrow[r, "T^{\mathsf{op}}"] & \mathcal{E}^{\mathsf{op}} \arrow[r, bend left, "\widetilde{\mathcal{U}}"] \arrow[r, bend right, "\mathcal{U}"'] \arrow[r, phantom, "\Downarrow \mathsf{ty}"] & \mathsf{Set}. \end{tikzcd} \tag{2.17}$$

This observation motivates the following definition.

2.2.18. DEFINITION (relative natural model). Let $P : \mathcal{E} \to \mathcal{B}$ be a functor. Suppose that $\mathcal{B}$ has a specified terminal object $1_{\mathcal{B}}$ and that P has a fibrewise natural model structure given by $T : \mathcal{B} \to \mathcal{E}$ and $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $\widehat{\mathcal{E}}$. Then we say that there is a *relative natural model* on $\mathcal{B}$ if the functor

$$\mathsf{ty} \circ T^{\mathsf{op}} : \widetilde{\mathcal{U}} \circ T^{\mathsf{op}} \to \mathcal{U} \circ T^{\mathsf{op}}$$

as in (2.17), is a representable natural transformation in $\widehat{\mathcal{B}}$. When this is the case, the represented pullbacks of $\mathsf{ty} \circ T^{\mathsf{op}}$ are denoted

$$\begin{tikzcd} B.\sigma \arrow[r, "w_\sigma"] \arrow[d, "q_\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \circ T^{\mathsf{op}} \arrow[d, "\mathsf{ty} \circ T^{\mathsf{op}}"] \\ B \arrow[r, "\sigma"'] & \mathcal{U} \circ T^{\mathsf{op}}. \end{tikzcd} \tag{2.18}$$

2.2.19. *Terminology.* A map $q : B' \to B$ in $\mathcal{B}$ is said to be a *display map* if it is a specified representative of $\mathsf{ty} \circ T^{\mathrm{op}}$ (as in Terminology 2.1.15).

2.2.20. *Remark.* In the situation of Definition 2.2.18, we have the following intended interpretation of judgements in the type theory associated with the first context zone:

$$\begin{aligned} \Delta \vdash \sigma \text{ type} &\quad \leftrightsquigarrow \quad \sigma \in \mathcal{U} \circ T^{\mathrm{op}}(B) \\ \Delta \vdash t : \sigma &\quad \leftrightsquigarrow \quad t \in \widetilde{\mathcal{U}} \circ T^{\mathrm{op}}(B) \text{ and } \sigma = (\mathsf{ty} \circ T^{\mathrm{op}}) \circ t \\ \Delta.\sigma \vdash w_\sigma : \sigma[q_\sigma] &\quad \leftrightsquigarrow \quad (\mathsf{ty} \circ T^{\mathrm{op}}) \circ w_\sigma = \sigma \circ q_\sigma \end{aligned}$$

By Observation 2.2.17, we also have an equivalent intended interpretation of dual-context judgements:

$$\begin{aligned} \Delta \mid \cdot \vdash \sigma \text{ type} &\quad \leftrightsquigarrow \quad \sigma \in \mathcal{U} \circ T^{\mathrm{op}}(B) \\ \Delta \mid \cdot \vdash t : \sigma &\quad \leftrightsquigarrow \quad t \in \widetilde{\mathcal{U}} \circ T^{\mathrm{op}}(B) \text{ and } \sigma = (\mathsf{ty} \circ T^{\mathrm{op}}) \circ t \\ \Delta.\sigma \mid \cdot \vdash w_\sigma : \sigma[q_\sigma] &\quad \leftrightsquigarrow \quad (\mathsf{ty} \circ T^{\mathrm{op}}) \circ w_\sigma = \sigma \circ q_\sigma \end{aligned}$$

Continuing the internal language notation, we have the following expressions associated with $\mathcal{B}$:

$$\begin{aligned} B \vdash \sigma \text{ type} &\quad \leftrightsquigarrow \quad \sigma \in \mathcal{U} \circ T^{\mathrm{op}}(B) \\ B \vdash t : \sigma &\quad \leftrightsquigarrow \quad t \in \widetilde{\mathcal{U}} \circ T^{\mathrm{op}}(B) \text{ and } \sigma = (\mathsf{ty} \circ T^{\mathrm{op}}) \circ t \\ B.\sigma \vdash w_\sigma : \sigma[q_\sigma] &\quad \leftrightsquigarrow \quad (\mathsf{ty} \circ T^{\mathrm{op}}) \circ w_\sigma = \sigma \circ q_\sigma \end{aligned}$$

and the following expressions associated with $\mathcal{E}$:

$$\begin{aligned} T(B) \vdash_B \sigma \text{ type} &\quad \leftrightsquigarrow \quad \sigma \in \mathcal{U} \circ T^{\mathrm{op}}(B) \\ T(B) \vdash_B t : \sigma &\quad \leftrightsquigarrow \quad t \in \widetilde{\mathcal{U}} \circ T^{\mathrm{op}}(B) \text{ and } \sigma = (\mathsf{ty} \circ T^{\mathrm{op}}) \circ t \\ T(B.\sigma) \vdash_{B.\sigma} w_\sigma : \sigma[q_\sigma] &\quad \leftrightsquigarrow \quad (\mathsf{ty} \circ T^{\mathrm{op}}) \circ w_\sigma = \sigma \circ q_\sigma \end{aligned}$$

Weakening of the context

Standard context weakening will be validated in our fibred model for the same reason as for standard type theory. Unique to dual-context type theory, it is possible to weaken the first context zone, as in the following rule:

$$\frac{\Delta \mid \Gamma \vdash \tau \text{ type} \qquad \Delta \mid \cdot \vdash \sigma \text{ type} \qquad r : \Delta.\sigma \mid \Gamma \to \Delta \mid \Gamma}{\Delta.\sigma \mid \Gamma \vdash \tau[r] \text{ type}}\ \textsc{ctx/wk/1st}$$

Using the internal language notation we have developed in Remarks 2.2.10, 2.2.16 and 2.2.20, this corresponds to showing the following rule is valid:

$$\frac{E \vdash_B \tau \text{ type} \qquad T(B) \vdash_B \sigma \text{ type} \qquad r : E' \to E}{E' \vdash_{B.\sigma} \tau[r] \text{ type}}$$

where $P(E) = B$ and E' is an object in $\mathcal{E}$ with $P(E') = B.\sigma$. That is, $r : E' \to E$ is a map between objects in different fibres. To get such a map $r : E' \to E$, take the pullback in the base category of the representable natural transformation along $\sigma : B \to \mathcal{U} \circ T^{\mathrm{op}}$, as in (2.18):

$$\begin{tikzcd} B.\sigma \arrow[r, "w_\sigma"] \arrow[d, "q_\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \circ T^{\mathrm{op}} \arrow[d, "\mathsf{ty} \circ T^{\mathrm{op}}"] \\ B \arrow[r, "\sigma"'] & \mathcal{U} \circ T^{\mathrm{op}}. \end{tikzcd}$$

If, for the display map $q_\sigma : B.\sigma \to B$ in $\mathcal{B}$, there is a specified choice of cartesian lift to E, denoted

$$\begin{tikzcd} E{*}\sigma \arrow[r, dashed, "r_{E,\sigma}"] \arrow[d, dotted, no head] & E \arrow[d, dotted, no head] & \mathcal{E} \arrow[d, "P"] \\ B.\sigma \arrow[r, "q_\sigma"] & B & \mathcal{B} \end{tikzcd}$$

then the rule is validated by the composition

$$E{*}\sigma \xrightarrow{r_{E,\sigma}} E \xrightarrow{\tau} \mathcal{U}.$$

Thus, for the functor $P : \mathcal{E} \to \mathcal{B}$ to model dual-context type theory, we require that there are cartesian lifts of display maps in $\mathcal{B}$. We introduce the following terminology for these lifts.

2.2.21. *Terminology.* A map $r : E' \to E$ is a *horizontal display map* if it is a cartesian map with respect to P, and its projection $P(r)$ is a display map in $\mathcal{B}$.

We are now prepared to gather the categorical analysis from this section into a notion of model of dual-context type theory.

2.3 FIBRED NATURAL MODEL OF DUAL-CONTEXT TYPE THEORY

In this section, we summarise the development in the previous section with the definition of *fibred natural model* (Definition 2.3.1). We also collect together some important terminology from Section 2.2, consider the extent to which a fibred natural model $P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration, and define morphisms of fibred natural models. Finally, we give an alternative axiomatisation of fibred natural model that will be useful for Chapter 3.

2.3.1. DEFINITION (Fibred natural model). A *fibred natural model of dual-context type theory* is a functor $P : \mathcal{E} \to \mathcal{B}$ and a representable natural transformation $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $\widehat{\mathcal{E}}$ such that:

(i) the base category $\mathcal{B}$ has a specified terminal object $1_{\mathcal{B}}$

(ii) the functor P has a right adjoint right inverse $T : \mathcal{B} \to \mathcal{E}$

(iii) the representable natural transformation $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $\widehat{\mathcal{E}}$ has fibrewise pullbacks, meaning that for the specified pullback for an object E and arrow $\sigma : E \to \mathcal{U}$,

$$\begin{tikzcd} E.\sigma \arrow[r, "v_\sigma"] \arrow[d, "p_\sigma"'] & \widetilde{\mathcal{U}} \arrow[d, "\mathsf{ty}"] \\ E \arrow[r, "\sigma"'] & \mathcal{U}, \end{tikzcd} \tag{2.19}$$

the map p_σ is in the fibre $\mathcal{E}_{P(E)}$, that is, $P(p_\sigma) = \mathrm{id}_{P(E)}$

(iv) the morphism $\mathsf{ty}\circ T^{\mathrm{op}} : \widetilde{\mathcal{U}}\circ T^{\mathrm{op}} \to \mathcal{U}\circ T^{\mathrm{op}}$ in $\widehat{\mathcal{B}}$ is a representable natural transformation, with pullbacks denoted

$$\begin{tikzcd} B.\sigma \arrow[r, "w_\sigma"] \arrow[d, "q_\sigma"'] & \widetilde{\mathcal{U}}\circ T^{\mathrm{op}} \arrow[d, "\mathsf{ty}\circ T^{\mathrm{op}}"] \\ B \arrow[r, "\sigma"'] & \mathcal{U}\circ T^{\mathrm{op}} \end{tikzcd} \tag{2.20}$$

(v) for each $\sigma \in \mathcal{U}(T(P(E)))$, the map $q_\sigma : P(E).\sigma \to P(E)$ of the pullback

$$\begin{tikzcd} P(E).\sigma \arrow[r, "w_\sigma"] \arrow[d, "q_\sigma"'] & \widetilde{\mathcal{U}}\circ T^{\mathrm{op}} \arrow[d, "\mathsf{ty}\circ T^{\mathrm{op}}"] \\ P(E) \arrow[r, "\sigma"'] & \mathcal{U}\circ T^{\mathrm{op}} \end{tikzcd}$$

in $\mathcal{B}$ has a specified choice of P-cartesian lift to E, denoted

$$E{*}\sigma \xrightarrow{r_{E,\sigma}} E.$$

2.3.2. *Terminology.* Alternatively, we say that a functor $P : \mathcal{E} \to \mathcal{B}$ *admits* a fibred natural model of dual-context type theory if there exists a representable natural transformation $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $\widehat{\mathcal{E}}$ and together P and ty satisfy the conditions of Definition 2.3.1.

We summarise the following terminology from Section 2.2, which will be important in Chapters 3 and 4.

2.3.3. *Terminology.* Suppose $P : \mathcal{E} \to \mathcal{B}$ admits a fibred natural model, given by the data

$$(1_{\mathcal{B}}, T : \mathcal{B} \to \mathcal{E}, \mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}).$$

Then:

(a) A vertical map in $\mathcal{E}$ is said to be a *vertical display map* (resp. *vertical representative*) if it is a display map (resp. *representative*) associated with ty.

(b) A map $r : E' \to E$ in $\mathcal{E}$ is a *horizontal display map* if it is a cartesian map with respect to P, and its projection $P(r)$ is a display map in $\mathcal{B}$.

(c) A map $q : B' \to B$ in $\mathcal{B}$ is said to be a *display map* if it is a represented pullback of $\mathsf{ty}\circ T^{\mathrm{op}}$.

2.3.4. *Remark.* Vertical display maps model extension of the second part of the context, while horizontal display maps model extension of the first part of the context.

2.3.5. *Remark.* A functor $P : \mathcal{E} \to \mathcal{B}$ admiting a fibred natural model is not assumed to be a Grothendieck fibration. Axiom (v) is a milder condition, motivated by the fact that it will hold in the example of a category with an idempotent comonad considered in Chapter 3. To understand the extent to which Axiom (v) ensures our model is a Grothendieck fibration, let E be an object in $\mathcal{E}$ that can be obtained by a finite sequence of vertical display maps in $\mathcal{E}$, as in

$$E = E_m \twoheadrightarrow E_{m-1} \twoheadrightarrow \cdots \twoheadrightarrow E_1 \twoheadrightarrow E_0 = T(P(E)).$$

Then any map $f : B \to P(E)$ in $\mathcal{B}$ admits a cartesian lift $\chi_{E,f} : E' \to E$. Furthermore, this object E' can also be obtained as a sequence of vertical display maps. Therefore, if every object in $\mathcal{E}$ can be obtained as a sequence of vertical display maps, then $P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration. This is the case in, for example, a term model, where all the objects are syntactic contexts constructed via a finite sequence of context extensions from the empty context.

2.3.6. *Remark.* Our notion of fibred natural model relates to the general frameworks of Gratzer et al. [GKNB20; Gra23] and Uemura [Uem21; Uem23a] as follows. *Fibred natural model* coincides with Gratzer's notion of *model of dual-context type theory without modal types* [Gra23, Definition 5.4.7, pp. 77], albeit with a different axiomatisation resulting from the modularity of our development. Gratzer's notion is compared to that obtained from the general framework of *multimodal type theory* (MTT) [GKNB20] in [Gra23, Theorem 7.4.2]. MTT is a general framework for modal type theory that takes a *mode theory*—an abstract description of a collection of modalities—to a modal type theory. A model is inherited by virtue of MTT being defined as a generalised algebraic theory, which is reformulated for convenience in terms of natural models in [Gra23, Section 7.1]. While our notion can be related to MTT in this way, our particular description of a model is more suited to our applications in subsequent chapters. The general framework of [Uem21] based on *categories with representable maps* does not include type theories with restrictions on the context, as is the case for dual-context type theory. However, a fibred version incorporating mode theories is in development [Uem23b].

Fibred natural models and strict structure preserving morphisms form a category, with morphisms given as follows.

2.3.7. DEFINITION. A *morphism of fibred natural models* $(\mathcal{E}, \mathcal{B}, P, T)$ and $(\mathcal{E}', \mathcal{B}', P', T')$ is a pair of functors $(F : \mathcal{E} \to \mathcal{E}', G : \mathcal{B} \to \mathcal{B}')$ such that:

(i) the specified terminal object is preserved:

$$G(1_{\mathcal{B}}) = 1_{\mathcal{B}'}$$

(ii) fibrewise terminal objects are preserved:

$$F(T(B)) = T'(G(B))$$

for all objects B in $\mathcal{B}$

(iii) specified pullbacks of the locally representable maps ty and ty' are preserved:

$$F(E.\sigma) = F(E).F(\sigma)$$
$$F(p_\sigma) = p_{F(\sigma)}$$
$$F(v_\sigma) = v_{F(\sigma)}$$

(iv) specified pullbacks of the locally representable maps $T^{op} \circ \mathsf{ty}$ and $T'^{op} \circ \mathsf{ty}'$ are preserved:

$$G(B.\sigma) = G(B).G(\sigma)$$
$$G(q_\sigma) = q_{G(\sigma)}$$
$$G(w_\sigma) = w_{G(\sigma)}$$

(v) cartesian morphisms are weakly preserved, that is, for each $r_\sigma : E{*}\sigma \to E$, the map

$$H(r_\sigma) : H(E{*}\sigma) \to H(E)$$

is a P'-cartesian morphism.

Comprehension adjoint

In this section, we present an alternative to Axiom (iv) in Definition 2.3.1. Where Axiom (iv) natively assumes that the base category has a natural model structure, we may instead use the machinery of the natural model structure in the total category and transport it to the base category. This is the content of the next proposition, which will be used to prove the main theorem in Chapter 3—that a category with an idempotent comonad admits a fibred natural model (Theorem 3.2.8).

2.3.8. PROPOSITION. *Let $P : \mathcal{E} \to \mathcal{B}$ be a functor satisfying Axioms (i)-(iii) of a fibred natural model (Definition 2.3.1), with data given by $(1_{\mathcal{B}}, T : \mathcal{B} \to \mathcal{E}, \mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U})$. Suppose that T has a right adjoint left inverse $S : \mathcal{E} \to \mathcal{B}$, as in*

$$\begin{tikzcd} \mathcal{E} \arrow[d, "P"'] \arrow[d, phantom, shift left=4, "\dashv"] \arrow[d, bend right=20, leftarrow, shift left=6, "T"] \arrow[d, phantom, shift left=9, "\dashv"] \arrow[d, bend left=20, shift left=11, "S"] \\ \mathcal{B}. \end{tikzcd}$$

Then the restriction of ty by T^{op},

$$ty{\circ}T^{op} : \widetilde{\mathcal{U}}{\circ}T^{op} \to \mathcal{U}{\circ}T^{op},$$

is a representable natural transformation in $\widehat{\mathcal{B}}$.

Proof. Let B be an object in $\mathcal{B}$ and let $\sigma \in \mathcal{U}{\circ}T^{\mathsf{op}}(B)$. To show that $\mathsf{ty}{\circ}T^{\mathsf{op}}$ is representable, we need to show that it admits a represented pullback for B and σ, that is, an object Q in $\mathcal{B}$, a map $q : Q \to B$ in $\mathcal{B}$, and an element $w \in \widetilde{\mathcal{U}}{\circ}T^{\mathsf{op}}(Q)$ assembling into the following pullback:

$$\begin{tikzcd} Q \arrow[r, "w"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}}{\circ}T^{\mathsf{op}} \arrow[d, "\mathsf{ty}"] \\ B \arrow[r, "\sigma"'] & \mathcal{U}{\circ}T^{\mathsf{op}}. \end{tikzcd} \tag{2.21}$$

By Observation 2.2.17, $\sigma \in \mathcal{U}{\circ}T^{\mathsf{op}}(B)$ is isomorphic to both the above map and a map $\sigma : T(B) \to \mathcal{U}$ in $\widehat{\mathcal{E}}$. The representability of ty means there is a represented pullback for $T(B)$ in $\mathcal{E}$ and $\sigma \in \mathcal{U}(T((B))$, denoted

$$\begin{tikzcd} E \arrow[r, "v"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "\mathsf{ty}"] \\ T(B) \arrow[r, "\sigma"'] & \mathcal{U}. \end{tikzcd}$$

Applying the functor S to the map $p : E \to T(B)$ in $\mathcal{E}$

$$S(E) \xrightarrow{S(p)} ST(B)$$

and noting that $ST(B) = B$ provides a candidate for the object Q and map $q : Q \to B$ in (2.21). To get a candidate for $w : S(E) \to \widetilde{\mathcal{U}}{\circ}T^{\mathsf{op}}$, we use the counit of the adjunction $T \dashv S$, namely $\varepsilon : TS \Rightarrow \mathsf{id}_{\mathcal{E}}$. The map

$$TS(E) \xrightarrow{\varepsilon_E} E \xrightarrow{v} \widetilde{\mathcal{U}}$$

in $\widehat{\mathcal{E}}$ corresponds to an object $\upsilon[\varepsilon_E] \in \widetilde{\mathcal{U}}(TS(E))$, which also corresponds to a map

$$S(E) \xrightarrow{\upsilon[\varepsilon_E]} \widetilde{\mathcal{U}} \circ T^{\mathrm{op}}$$

in $\widehat{\mathcal{B}}$. Our candidate for the desired pullback square is then

$$\begin{tikzcd} S(E) \arrow[r, "\upsilon[\varepsilon_E]"] \arrow[d, "S(p)"'] & \widetilde{\mathcal{U}} \circ T^{\mathrm{op}} \arrow[d, "\mathsf{ty} \circ T^{\mathrm{op}}"] \\ B \arrow[r, "\sigma"'] & \mathcal{U} \circ T^{\mathrm{op}}. \end{tikzcd} \tag{2.22}$$

To see that this commutes,

$$\begin{tikzcd} TS(E) \arrow[r, "\varepsilon_E"] \arrow[d, "TS(p)"'] & E \arrow[r, "\upsilon"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "\mathsf{ty}"] \\ TST(B) \arrow[r, equal, "\varepsilon_{TB}"'] & T(B) \arrow[r, "\sigma"'] & \mathcal{U}. \end{tikzcd}$$

To see that the square in (2.22) is a pullback, consider an object X in $\widehat{\mathcal{B}}$ and maps $t : \widetilde{\mathcal{U}} \circ T^{\mathrm{op}}$, $q : X \to B$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "q"'] & & \\ & S(E) \arrow[r, "\upsilon"] \arrow[d, "S(p)"'] & \widetilde{\mathcal{U}} \circ T^{\mathrm{op}} \arrow[d, "\mathsf{ty} \circ T^{\mathrm{op}}"] \\ & B \arrow[r, "\sigma"'] & \mathcal{U} \circ T^{\mathrm{op}}. \end{tikzcd}$$

Then we have the following commutative diagram in $\widehat{\mathcal{E}}$,

$$\begin{tikzcd} TX \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "T(q)"'] \arrow[dr, dashed, "T(q).t"] & & \\ & E \arrow[r, "\upsilon"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "\mathsf{ty}"] \\ & TB \arrow[r, "\sigma"'] & \mathcal{U}, \end{tikzcd}$$

in which the pullback property of the square induces a unique map $T(q).t : T(X) \to E$. So there is a bijection between pairs (q, t) and maps $X \to S(E)$. □

2.3.9. *Observation.* In order to later apply Proposition 2.3.8, we record here a universal property for right adjoint left inverses analogous to Observation 2.2.8. Namely, a functor $T : \mathcal{B} \to \mathcal{E}$ has a right adjoint left inverse just if for all objects E in $\mathcal{E}$ and maps $f : B \to S(E)$ in $\mathcal{B}$, there exists a unique map $\hat{f} : T(B) \to E$ with $S(\hat{f}) = f$.

In the next chapter, we will consider an example of a fibred natural model admitted by a category with an idempotent comonad. This will form a bridge to Chapter 4, where we specialise further to a presheaf category and a specific idempotent comonad, and give a complete presentation of an internal dual-context type theory.

3 Fibred models from a category with an idempotent comonad

Introduction

In Chapter 2, we saw abstractly how to model the features of a dual-context type theory, culminating in the notion of fibred natural model. More concretely, the intended models of crisp type theory are *local toposes*, specifically presheaf categories and a particular idempotent comonad. In this chapter, we build a bridge from abstract models to these specific models by considering the intermediate step of any category with an idempotent comonad. At this intermediate level of abstraction, we establish two main results.

The first result (Theorem 3.2.8) says that a category with a stable class of maps and an idempotent comonad, plus some other routine assumptions for admitting a model of standard dependent type theory, gives rise to a fibred natural model (Definition 2.3.1). This will be applied to a presheaf category in Chapter 4 to then extract crisp type theory as an internal language. The second result (Theorem 3.3.5) says that with the further assumption of a classifier for the stable class of maps, there are classifiers for stable classes of maps in the total and base categories of the fibred model, and that these type universes are related in the same way as the type universes in a fibred natural model (Axiom (iv)). The consequence is that such a category admits a fibred model of dual-context type theory via categories with classified stable maps, that is, a fibred version of the other flavour of model introduced in Chapter 2 and used for the applications in Chapter 4 onwards.

To this end, Section 3.1 recalls the relevant theory of idempotent comonads, including a convenient result that means we can assume, without loss of generality, that the idempotent comonad on our category is strict. The subsequent sections (Section 3.2 and Section 3.3) build the main results (Theorem 3.2.8 and Theorem 3.3.5) respectively.

3.1 PRELIMINARIES: IDEMPOTENT COMONADS

Let $\mathcal{C}$ be a small category. In this section, we recall equivalent characterisations of the standard notion of an idempotent comonad and a strict version thereof. We also recall the precise sense in which an idempotent comonad can be replaced by an equivalent strictly idempotent comonad. This will allow us to work with (more convenient) strict idempotent comonads when we consider fibred models arising from a category with an idempotent comonad in Section 3.2 and beyond.

We begin with the definition of a comonad.

3.1.1. DEFINITION. A *comonad* on a category $\mathcal{C}$ consists of an endofunctor $G : \mathcal{C} \to \mathcal{C}$, a *counit* natural transformation $\varepsilon : G \Rightarrow \mathrm{id}_{\mathcal{C}}$, and a *comultiplication* natural transformation $\nu : G \Rightarrow G^2$ such that the following diagrams commute:

$$\begin{tikzcd} & G \arrow[dl, Rightarrow, "\mathrm{id}_G"'] \arrow[d, Rightarrow, "\nu"] \arrow[dr, Rightarrow, "\mathrm{id}_G"] & \\ G & G^2 \arrow[l, Rightarrow, "\varepsilon G"] \arrow[r, Rightarrow, "G\varepsilon"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\nu"] \arrow[d, Rightarrow, "\nu"'] & G^2 \arrow[d, Rightarrow, "G\nu"] \\ G^2 \arrow[r, Rightarrow, "\nu G"'] & G^3. \end{tikzcd}$$

We now give definitions of idempotent comonad and strictly idempotent comonad that will be justified by Proposition 3.1.3.

3.1.2. DEFINITION. Given an endofunctor $G : \mathcal{C} \to \mathcal{C}$ and a natural transformation $\varepsilon : G \Rightarrow \mathrm{id}_{\mathcal{C}}$, we say that (G, ε) is:

(i) an *idempotent comonad* if $G\varepsilon : G^2 \Rightarrow G$ and $\varepsilon G : G^2 \Rightarrow G$ are both isomorphisms

(ii) a *strictly idempotent comonad* if $G\varepsilon : G^2 \Rightarrow G$ and $\varepsilon G : G^2 \Rightarrow G$ are both identities.

The next proposition relates this definition of an idempotent comonad to the standard definition of a comonad given previously.

3.1.3. PROPOSITION. *If (G, ε) is an idempotent comonad (or strictly idempotent comonad) then $G\varepsilon = \varepsilon G$, and their common inverse $\nu : G \Rightarrow G^2$ is the comultiplication of a comonad (G, ε, ν). Conversely, if the comultiplication of a comonad (G, ε, ν) is an isomorphism (respectively identity) then (G, ε) is an idempotent comonad (respectively strictly idempotent comonad).*

The following corollary is the simplified notion of coalgebra that occurs when a comonad is idempotent.

3.1.4. COROLLARY. *Suppose (G, ε) is an idempotent comonad on $\mathcal{C}$. Then an object C in $\mathcal{C}$ supports a coalgebra structure $\alpha : C \to GC$ if and only if ε_C is an isomorphism, in which case $\alpha = \varepsilon_C^{-1}$.*

3.1.5. *Notation.* (a) Let $\mathcal{C}^G$ denote the full subcategory of $\mathcal{C}$ spanning those objects C with ε_C an isomorphism.

(b) Let $\mathrm{Fix}_{\mathcal{C}}(G)$ denote the full subcategory of $\mathcal{C}$ spanning those objects C with ε_C an identity.

3.1.6. *Observation.* The category $\mathcal{C}^G$ is isomorphic to the category of Eilenberg-Moore coalgebras for G. The Eilenberg-Moore category consists of those objects C in $\mathcal{C}$ supporting a coalgebra structure, and coalgebra maps between them. Suppose C and D both support coalgebra structures, then as a consequence of Corollary 3.1.4, any arrow $f : C \to D$ in $\mathcal{C}$ lifts to a coalgebra map between them, so this category is isomorphic to the full subcategory of $\mathcal{C}$ spanning those objects C with a coalgebra structure, or equivalently, with ε_C an isomorphism.

The next observation describes the relationship between the categories $\mathcal{C}^G$ and $\mathcal{C}$.

3.1.7. *Observation.* Let (G, ε) be an idempotent comonad. Then G factors uniquely through the full subcategory inclusion $I : \mathcal{C}^G \hookrightarrow \mathcal{C}$ via a functor $G' : \mathcal{C} \to \mathcal{C}^G$ right adjoint to I, with counit

$$I \circ G' = G \Longrightarrow \mathsf{id}_{\mathcal{C}}$$

equal to ε.

There is another right adjoint to the inclusion functor.

3.1.8. LEMMA. *Suppose $\mathcal{D}$ is a coreflective subcategory of $\mathcal{C}$. Then there exists a strictly idempotent comonad $(\overline{G}, \overline{\varepsilon})$ on $\mathcal{C}$ such that $\mathcal{D} = \mathsf{Fix}_{\mathcal{C}}(\overline{G})$.*

3.1.9. COROLLARY. *If (G, ε) is an idempotent comonad then there exists a strictly idempotent comonad $(\overline{G}, \overline{\varepsilon})$ such that $\mathcal{C}^G = \mathsf{Fix}_{\mathcal{C}}(\overline{G})$.*

The last corollary tells us that we may replace an idempotent comonad by an equivalent strictly idempotent comonad with the same coalgebras. Working with a strictly idempotent comonad will simplify the development of this chapter.

3.2 FIBRED NATURAL MODELS FROM AN IDEMPOTENT COMONAD

Let $\mathcal{C}$ be a small category with a specified terminal object $1_{\mathcal{C}}$, a stable class of maps S in the sense of Definition 2.1.3, and an idempotent comonad $(\flat, \varepsilon : \flat \Rightarrow \mathsf{id}_{\mathcal{C}})$ that preserves the terminal object, the class S, and pullbacks of maps in S. In light of Section 3.1, we will assume without loss of generality that $\flat$ is a strictly idempotent comonad. In this section, we prove that $\mathcal{C}$ admits a fibred natural model (Theorem 3.2.8).

To show how such a category admits a fibred natural model, we begin by specifying the data—the total category, base category, and functors between them—as well as how each of these categories inherits a stable class of maps from the class S in $\mathcal{C}$. We then use results from the previous chapter to show that these data satisfy the axioms of a fibred natural model. Specifically, Observations 2.2.8 and 2.3.9 are used to show that the relevant functors are a right adjoint right inverse and a right adjoint left inverse respectively, and Example 2.1.20 is followed to construct a representable natural transformation on the total category from the stable class of maps.

Fibred natural model: the data

3.2.1. *Notation.* (a) Let $\mathcal{C}_\flat$ denote the full subcategory of $\mathcal{C}$ spanned by objects B with ε_B the identity on B (that is, $\mathsf{Fix}_{\mathcal{C}}(\flat)$ from Notation 3.1.5).

(b) Let $\mathcal{C}\downarrow\mathcal{C}_\flat$ denote the comma category with objects

$$C_\bullet = (C, B, c : C \to B),$$

where C is an object in $\mathcal{C}$, B is an object in $\mathcal{C}_\flat$, and c is an arrow in $\mathcal{C}$. An arrow

$$(f, g) : C_\bullet \to C'_\bullet$$

consists of an arrow f in $\mathcal{C}$ and g in $\mathcal{C}_\flat$ such that the following square commutes:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "c"'] & C' \arrow[d, "c'"] \\ B \arrow[r, "g"'] & B'. \end{tikzcd}$$

3.2.2. *Remark.* The fibred natural model structure will come from taking $\mathcal{C}{\downarrow}\mathcal{C}_\flat$ as the total category, $\mathcal{C}_\flat$ as the base category, and functors between them

$$\begin{tikzcd} \mathcal{C}{\downarrow}\mathcal{C}_\flat \arrow[d, "\mathsf{cod}"', bend right] \arrow[d, phantom, "\dashv"] \\ \mathcal{C}_\flat, \arrow[u, bend right, "T"'] \end{tikzcd} \quad \dashv S$$

where

- cod is the codomain functor, given on objects and arrows by

$$\begin{aligned} \mathsf{cod}(C, B, c : C \to B) &= B \\ \mathsf{cod}(f, g) &= g \end{aligned}$$

- T is the terminal functor, given on an object D and an arrow $u : D \to D'$ in $\mathcal{C}_\flat$ by

$$\begin{aligned} T(D) &= (D, D, \mathsf{id}_D) \\ T(u) &= (u, u) \end{aligned}$$

- S is the comprehension functor, given on objects and arrows by

$$\begin{aligned} S(C, B, c : C \to B) &= \flat C \\ S(f, g) &= \flat f. \end{aligned}$$

To get natural models on the total and base categories, we will follow Example 2.1.20 to construct a representable natural transformation for a stable class of maps. The categories $\mathcal{C}{\downarrow}\mathcal{C}_\flat$ and $\mathcal{C}_\flat$ each inherit a stable class of maps from the stable class S in $\mathcal{C}$ as follows.

3.2.3. LEMMA (stable maps in $\mathcal{C}{\downarrow}\mathcal{C}_\flat$). *The comma category $\mathcal{C}{\downarrow}\mathcal{C}_\flat$ (Notation 3.2.1) inherits a stable class of maps from $\mathcal{C}$, namely those of the form*

$$(d, \mathsf{id}_B) : (C, B, c : C \to B) \to (C', B, c' : C' \to B),$$

with $d : C \twoheadrightarrow C'$ in S, corresponding to a commutative diagram

$$\begin{tikzcd} C \arrow[r, "d", ->>] \arrow[d, "c"'] & C' \arrow[d, "c'"] \\ B \arrow[r, equal] & B. \end{tikzcd}$$

We denote this class by $\mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat}$.

Proof. Fix a morphism $(d, \mathsf{id}_B) \in \mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat}$ and consider any morphism $(f : C'' \to C', g : B'' \to B)$ in $\mathcal{C}\downarrow\mathcal{C}_\flat$, as in the following diagram:

$$\begin{tikzcd}
 & & C \arrow[dl, "d"'] \arrow[d, "c"] \\
C'' \arrow[r, "f"] \arrow[d, "c''"'] & C' \arrow[d, "c'"'] & B \arrow[dl, equal] \\
B'' \arrow[r, "g"'] & B &
\end{tikzcd}$$

A pullback of the upper cospan exists because d is a stable map in $\mathcal{C}$, and there is the obvious pullback of the lower cospan, giving the following commutative diagram

$$\begin{tikzcd}
 & R \arrow[rr, "q"] \arrow[dl, "p"'] & & C \arrow[dl, "d"] \arrow[dd, "c"] \\
C'' \arrow[rr, "f"] \arrow[dd, "c''"'] & & C' & \\
 & B'' \arrow[rr] & & B \\
B'' \arrow[ur, equal] \arrow[rr, "g"'] & & B \arrow[ur, equal] \arrow[from=uu, crossing over, "c'"] &
\end{tikzcd}$$

in which each of the top face and bottom face are pullback diagrams in $\mathcal{C}$. By the universal property of the top face pullback square, there exists a unique map $r : R \to B''$ making the diagram commute:

$$\begin{tikzcd}
 & R \arrow[rr, "q"] \arrow[dl, "p"'] \arrow[dd, dashed] & & C \arrow[dl, "d"] \arrow[dd, "c"] \\
C'' \arrow[rr, "f"] \arrow[dd, "c''"'] & & C' & \\
 & B'' \arrow[rr] & & B \\
B'' \arrow[ur, equal] \arrow[rr, "g"'] & & B \arrow[ur, equal] \arrow[from=uu, crossing over, "c'"] &
\end{tikzcd}$$

The morphism $(p, \mathsf{id}_{B''})$ belongs to the class $\mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat}$, and it is straightforward to see that this forms a pullback in the comma category, by using the universal properties of the upper and lower pullback squares in $\mathcal{C}$. □

3.2.4. LEMMA (stable class of maps in $\mathcal{C}_\flat$). *The subcategory $\mathcal{C}_\flat$ (Notation 3.2.1) inherits a stable class of maps from $\mathcal{C}$, namely those maps $d : B \rightarrow B'$ in S such that $\flat d = d$. We denote this class by $S_{\mathcal{C}_\flat}$.*

Proof. The category $\mathcal{C}_\flat$ is a coreflective subcategory of $\mathcal{C}$, with right adjoint to the inclusion functor given by $\flat$,

$$\begin{tikzcd}
\mathcal{C}_\flat \arrow[r, hook, bend left] \arrow[r, phantom, "\perp"] & \mathcal{C}. \arrow[l, bend left, "\flat"]
\end{tikzcd}$$

The stability of the class $\mathsf{S}_{\mathcal{C}_\flat}$ under pullback then follows from the fact that right adjoints preserve limits. □

Fibred natural model: the axioms

We establish some results to prove the axioms of a fibred natural model hold.

3.2.5. LEMMA. *The terminal functor $T : \mathcal{C}_\flat \to \mathcal{C}{\downarrow}\mathcal{C}_\flat$, defined in Remark 3.2.2, is a right adjoint right inverse for cod* $: \mathcal{C}{\downarrow}\mathcal{C}_\flat \to \mathcal{C}_\flat$.

Proof. We prove this by showing that T satisfies the universal property of a right adjoint right inverse from Observation 2.2.8. Let $C_\bullet = (C, B, c : C \to B)$ be an object in $\mathcal{C}{\downarrow}\mathcal{C}_\flat$ and $w : \mathrm{cod}(C_\bullet) \to D$ be an arrow in $\mathcal{C}_\flat$, that is, $w : B \to D$. Morphisms from $C_\bullet$ to $T(D)$ are arrows $f : C \to D$ in $\mathcal{C}$ and $g : B \to D$ in $\mathcal{C}_\flat$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "c"'] & D \arrow[d, equal] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

commutes. The commutativity of this square says that such an arrow must have the form $(g \circ c, g)$, so there can only be one map $\widehat{w} : C_\bullet \to T(D)$ with $\mathrm{cod}(\widehat{w}) = w$, namely $\widehat{w} = (w \circ c, w)$. □

3.2.6. LEMMA. *The comprehension functor $S : \mathcal{C}{\downarrow}\mathcal{C}_\flat \to \mathcal{C}_\flat$, defined in Remark 3.2.2, is a right adjoint left inverse for T.*

Proof. We will show that S satisfies the universal property of a right adjoint left inverse from Observation 2.3.9. Let $C_\bullet = (C, B, c : C \to B)$ be an object in $\mathcal{C}{\downarrow}\mathcal{C}_\flat$ and $u : D \to S(C_\bullet)$ be an arrow in $\mathcal{C}_\flat$, that is,

$$u : D \to \flat C.$$

To see that there is a unique arrow $\hat{u} : T(D) \to C_\bullet$ with $S(\hat{u}) = u$, first note that $\hat{u}$ is given by maps $f : D \to C$ in $\mathcal{C}$ and $g : D \to B$ in $\mathcal{C}_\flat$ such that the diagram

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, equal] & C \arrow[d, "c"] \\ D \arrow[r, "g"'] & B \end{tikzcd}$$

commutes, and so is fully determined by a map $f : D \to C$. Let $f = \varepsilon_C \circ u$, as in

$$D \xrightarrow{u} \flat C \xrightarrow{\varepsilon_C} C.$$

Then $\hat{u} = (\varepsilon_C \circ u, c \circ \varepsilon_C \circ u)$ and applying S we have

$$S(\hat{u}) = \flat(\varepsilon_C \circ u) = \flat\varepsilon_C \circ \flat u.$$

But $\flat$ is assumed to be a strictly idempotent comonad, and so $\flat\varepsilon_C = \mathrm{id}_C$ by Definition 3.1.2. Furthermore, u is a map in $\mathcal{C}_\flat$, so $\flat u = u$. Therefore,

$$S(\hat{u}) = \mathrm{id}_C \circ \flat u = u$$

as required. □

3.2.7. LEMMA. *There is a fibrewise representable natural transformation in $\widehat{\mathcal{C}{\downarrow}\mathcal{C}_\flat}$, in the sense of Definition 2.2.11, with fibrewise representatives given by the class of maps $\mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat}$ from Lemma 3.2.3.*

Proof. Having established in Lemma 3.2.3 that $\mathcal{C}{\downarrow}\mathcal{C}_\flat$ has a stable class of maps $\mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat}$, we can follow Example 2.1.20 to obtain a representable natural transformation. For an object $C_\bullet = (C, B, c : C \to B)$ in $\mathcal{C}{\downarrow}\mathcal{C}_\flat$, we define $\mathcal{U}(C_\bullet)$ as the set consisting of tuples of the form

$$(C', C'', g : C \to C', h : B \to B', d : C'' \twoheadrightarrow B')$$

such that the following diagram commutes:

$$\begin{tikzcd}
 & C'' \arrow[d, -{Triangle[open]}, "d"'] \arrow[dd, bend left, "c''"] \\
C \arrow[r, "g"] \arrow[d, "c"'] & C' \arrow[d, "c'"'] \\
B \arrow[r, "h"'] & B'.
\end{tikzcd} \tag{3.1}$$

More succinctly, we can simply let

$$\mathcal{U}(C_\bullet) := \{((g,h),(d,\mathsf{id}_{B'})) \in \mathsf{mor}(\mathcal{C}{\downarrow}\mathcal{C}_\flat) \times \mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat} \mid \mathsf{cod}(g,h) = \mathsf{cod}(d,\mathsf{id}_{C''})\}.$$

We define $\widetilde{\mathcal{U}}(C_\bullet)$ as the set consisting of tuples of the form

$$(C', C'', e : C \to C'', f : B \to B', d : C'' \twoheadrightarrow C')$$

such that the following diagram commutes:

$$\begin{tikzcd}
 & C'' \arrow[d, -{Triangle[open]}, "d"'] \arrow[dd, bend left, "c''"] \\
C \arrow[ur, bend left, "e"] \arrow[d, "c"'] & C' \arrow[d, "c'"'] \\
B \arrow[r, "f"'] & B'.
\end{tikzcd} \tag{3.2}$$

More succinctly, we let

$$\widetilde{\mathcal{U}}(C_\bullet) := \{((e,f),(d,\mathsf{id}_{B'})) \in \mathsf{mor}(\mathcal{C}{\downarrow}\mathcal{C}_\flat) \times \mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat} \mid \mathsf{cod}(e,f) = \mathsf{dom}(d,\mathsf{id}_{C''})\}.$$

Components of the natural transformation are given by

$$\begin{array}{ccc} \widetilde{\mathcal{U}}(C_\bullet) & \xrightarrow{\quad \mathsf{ty}_{C_\bullet} \quad} & \mathcal{U}(C_\bullet) \\ \big((e,f),(d,\mathsf{id}_{B'})\big) & \longmapsto & \big((e \circ d, f),(d,\mathsf{id}_{B'})\big). \end{array}$$

The action of the presheaves is by precomposition in the first factor. That is, for an object $N_\bullet = (N, M, n : N \to M)$ and morphism $(s,t) : N_\bullet \to C_\bullet$, as in

$$\begin{tikzcd}
N \arrow[r, "s"] \arrow[d, "n"'] & C \arrow[d, "c"] \\
M \arrow[r, "t"'] & B,
\end{tikzcd}$$

we make the following definitions:

$$\begin{aligned}\widetilde{\mathcal{U}}(s,t)\big((e,f),(d,\mathrm{id}_B')\big) &:= \big((e\circ s, f\circ t),(d,\mathrm{id}_B')\big)\\ \mathcal{U}(s,t)\big((g,h),(d,\mathrm{id}_B')\big) &:= \big((g\circ s, h\circ t),(d,\mathrm{id}_B')\big).\end{aligned}$$

By construction, the class of representative maps associated with $\mathsf{ty} : \widetilde{\mathcal{U}} \to \mathcal{U}$ is the class $\mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat}$. These are vertical maps with respect to the functor $\mathsf{cod} : \mathcal{C}\downarrow\mathcal{C}_\flat \to \mathcal{C}_\flat$ from Remark 3.2.2. Therefore, the representatives are in fact fibrewise representatives. □

We can now prove the main theorem: that $\mathcal{C}$ gives rise to a fibred natural model via the idempotent comonad.

3.2.8. THEOREM. *Let $\mathcal{C}$ be a small category with a specified terminal object $1_{\mathcal{C}}$, a stable class of maps S, and an idempotent comonad $(\flat, \varepsilon : \flat \Rightarrow id_{\mathcal{C}})$ that preserves the terminal object, stable maps, and pullbacks of stable maps. Then $\mathcal{C}$ gives rise to a fibred natural model via the functor*

$$cod : \mathcal{C}\downarrow\mathcal{C}_\flat \to \mathcal{C}_\flat$$

specified in Notation 3.2.1 and Remark 3.2.2.

Proof. Without loss of generality, we will assume that $\flat$ is a strictly idempotent comonad; if this is not the case, we can replace $(\flat,\varepsilon)$ by a $(\overline{\flat},\overline{\varepsilon})$, as in Corollary 3.1.8. We now address each axiom of Definition 2.3.1.

(i) The category $\mathcal{C}$ has a specified terminal object that is preserved by $\flat$, so $\flat 1$ is terminal in $\mathcal{C}$. The component of the counit $\varepsilon_{\flat 1} : \flat\flat 1 \to \flat 1$ is an identity since $\flat$ is a strictly idempotent comonad, so $\flat 1$ is in $\mathcal{C}_\flat$ and since it is terminal in $\mathcal{C}$, it is also terminal in $\mathcal{C}_\flat$.

(ii) The functor cod has a right adjoint right inverse T, by Lemma 3.2.5.

(iii) There is a fibrewise representable natural transformation in $\widehat{\mathcal{C}\downarrow\mathcal{C}_\flat}$, by Lemma 3.2.7.

(iv) The map

$$\mathsf{ty}\circ T^{op} : \widetilde{\mathcal{U}}\circ T^{op} \to \mathcal{U}\circ T^{op},$$

with ty, $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ defined as in Lemma 3.2.7, is a representable map in $\mathcal{C}_\flat$ by applying Proposition 2.3.8 to the right adjoint left inverse to T established in Lemma 3.2.6.

(v) Let $q : B' \twoheadrightarrow B$ be a stable map in $\mathcal{C}_\flat$. We can lift q to an object $C_\bullet = (C, B, c)$ in $\mathcal{C}\downarrow\mathcal{C}_\flat$ living over B as follows. Take the pullback

$$\begin{tikzcd} B'\times_B C \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "c"] \\ B' \arrow[r, "q"'] & B. \end{tikzcd}$$

This exists and r is a stable map because q is a stable map. To show that this is cartesian, for any $(f, g) : C''_{\bullet} \to C_{\bullet}$ and $w : B'' \to B'$ in $\mathcal{C}_\flat$ such that $g = q \circ w$, as in

$$\begin{tikzcd}
 & C'' \arrow[dd] \arrow[drrr, bend left, "f"] & & \\
\mathcal{C}{\downarrow}\mathcal{C}_\flat \arrow[ddd] & & B' \times_B C \arrow[r, "r"] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[dd, "c"] \\
 & B'' \arrow[drr, bend left] & & \\
 & & B' \arrow[r, "q"'] & B \\
\mathcal{C}_\flat & B'' \arrow[dr, "w"'] \arrow[drr, bend left, "g"] & & \\
 & & B' \arrow[r, "q"'] & B
\end{tikzcd}$$

there exists a unique map $C''_{\bullet} \to C'_{\bullet}$ using the universal property of the pullback.

□

3.3 Fibred category with a classified stable class of maps from an idempotent comonad

In Chapter 2, we described models of dual-context type theory as fibred *natural models*, but remarked that the applications in Chapter 4 onwards build on [AGH24], which uses a different style of model—namely, a category with a classified stable class of maps. In the former style of model, the notion of universe of types, given by the representable natural transformation, lives in the category of presheaves on the category containing the class of maps that model dependent types. In the latter model, the universe of types is given by a classifier (Definition 2.1.7) and lives in the same category as the stable class. To move from a fibred natural model to the setting of a fibred version of a category with a classified class of stable maps, we recall Corollary 2.1.23. This says that given a classifier for the stable class of maps S, we obtain a representable natural transformation with representatives S by taking the Yoneda embedding of the classifier.

In this section, we show that if, in addition to the assumptions on $\mathcal{C}$ in Theorem 3.2.8, we ask that $\mathcal{C}$ has a classifier of stable maps, then there exist classifiers for the stable maps of the total category $\mathcal{C}{\downarrow}\mathcal{C}_\flat$ and the base category $\mathcal{C}_\flat$. Crucially, these classifiers are "appropriately relative" in the sense of Axiom (iv) of a fibred natural model: when we take the Yoneda embedding of the classifier in the total category and restrict it to the base category via the functor T in Remark 3.2.2, the resulting map is isomorphic to the Yoneda embedding of the classifier in the base category. This guarantees that the type theory of the base category and the type theory of the total category have the required relationship for modelling dual-context type theory.

We begin with the classifier inherited by the total category.

3.3.1. PROPOSITION. *Let $\mathcal{C}$ be a small category with a specified terminal object $1_{\mathcal{C}}$, a stable class of maps S, and an idempotent comonad $(\flat, \varepsilon : \flat \Rightarrow id_{\mathcal{C}})$ that preserves the terminal object, stable maps, and pullbacks of stable maps. Suppose that $\pi : \widetilde{V} \to V$ is a classifier for S, in the sense of Definition 2.1.7. Then there exists a classifier for the stable class of maps $\mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat}$ inherited by $\mathcal{C}\downarrow\mathcal{C}_\flat$ by Lemma 3.2.3, namely the arrow (π, id_1) given diagrammatically by*

$$\begin{tikzcd}
& \widetilde{V} \arrow[dl, "\pi"'] \arrow[d, "!_{\widetilde{V}}"] \\
V \arrow[d, "!_V"'] & 1 \\
1 \arrow[ur, equals] &
\end{tikzcd} \tag{3.3}$$

Proof. Firstly, the candidate classifier $(\pi, \mathrm{id}_1) : !_{\widetilde{V}} \to !_V$ does indeed belong to $\mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat}$ since it has the form specified in Lemma 3.2.3. To show that it is a classifier, for any map

$$(d, \mathrm{id}_B) : (C, B, c : C \to B) \to (C', B, c' : C \to B)$$

in $\mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat}$, as in

$$\begin{tikzcd}
& C \arrow[dl, "d"'] \arrow[d, "c"] \\
C' \arrow[d, "c'"'] & B \\
B \arrow[ur, equals] &
\end{tikzcd}$$

we need dashed arrows making the below diagram into a pullback:

$$\begin{tikzcd}
& C \arrow[dl, "d"'] \arrow[d] \arrow[rr, dashed] & & \widetilde{V} \arrow[dl, "\pi"] \arrow[d] \\
C' \arrow[d, "c'"'] \arrow[rr, dashed] & B \arrow[rr, dashed] & V \arrow[d] & 1 \\
B \arrow[ur, equals] \arrow[rr, dashed] & & 1 \arrow[ur, equals] &
\end{tikzcd}$$

Taking dashed arrows in the lower square to be the unique map from B to the terminal object gives a pullback square. Dashed arrows in the upper square are given by the fact that π classifies d in $\mathcal{C}$, so there exist maps $f : C \to \widetilde{V}$ and $g : C' \to V$ making the following square into a pullback:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "d"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{V} \arrow[d, "\pi"] \\
C' \arrow[r, "g"'] & V.
\end{tikzcd}$$

The front and back faces of the cube

$$\begin{tikzcd}
& C \arrow[rr, "f"] \arrow[dl, "d"'] \arrow[dd] & & \widetilde{V} \arrow[dl, "\pi"] \arrow[dd, "!"] \\
C' \arrow[rr, "g"] \arrow[dd, "c'"'] & & V \arrow[dd] & \\
& B \arrow[rr, "!"] \arrow[dl, equal] & & 1 \arrow[dl, equal] \\
B \arrow[rr, "!"] & & 1 &
\end{tikzcd}$$

each commute since the universal property of the terminal object means there must be unique maps from C to 1 and C' to 1. □

3.3.2. PROPOSITION. *Let $\mathcal{C}$ be a small category with a specified terminal object $1_{\mathcal{C}}$, a stable class of maps S, and an idempotent comonad $(\flat, \varepsilon : \flat \Rightarrow id_{\mathcal{C}})$ that preserves the terminal object, stable maps, and pullbacks of stable maps. Suppose that $\pi : \widetilde{V} \to V$ is a classifier for S, in the sense of Definition 2.1.7. Then there exists a classifier for the stable class of maps $S_{\mathcal{C}_\flat}$ inherited by $\mathcal{C}_\flat$ by Lemma 3.2.4, namely*

$$\flat\pi : \flat\widetilde{V} \to \flat V.$$

Proof. The map $\flat\pi$ belongs to $\mathsf{S}_{\mathcal{C}_\flat}$ since the strict idempotence of $\flat$ means $\flat(\flat\pi) = \flat\pi$.

To show that it is a classifier, for a map $d : B \twoheadrightarrow B' \in \mathsf{S}_{\mathcal{C}_\flat}$, there is the following pullback square

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "d"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{V} \arrow[d, "\pi"] \\
B' \arrow[r, "g"'] & V
\end{tikzcd}$$

by the fact that π classifies d in $\mathcal{C}$. We use this to get the following pullback square in $\mathcal{C}_\flat$

$$\begin{tikzcd}
B \arrow[r, "\flat f"] \arrow[d, "d"'] \arrow[dr, phantom, "\lrcorner", very near start] & \flat\widetilde{V} \arrow[d, "\flat\pi"] \\
B' \arrow[r, "\flat g"'] & \flat V.
\end{tikzcd}$$

□

We now record that these classifiers give rise to representable natural transformations.

3.3.3. LEMMA. *The Yoneda embedding of the classifier (π, id_1) for the stable class $S_{\mathcal{C}\downarrow\mathcal{C}_\flat}$ in $\mathcal{C}\downarrow\mathcal{C}_\flat$ (Proposition 3.3.1) is a representable natural transformation with class of representatives given by $S_{\mathcal{C}\downarrow\mathcal{C}_\flat}$. The Yoneda embedding of the classifier $\flat\pi$ for the stable class $S_{\mathcal{C}_\flat}$ in $\mathcal{C}_\flat$ (Proposition 3.3.2) is a representable natural transformation, with class of representatives given by the class of stable maps $S_{\mathcal{C}_\flat}$.*

Proof. Follows immediately from Corollary 2.1.23. □

Finally, we show that the representable natural transformations from this lemma are related as for a fibred natural model. This guarantees that the type theory of the base category and the type theory of the total category, arising from the model of dependent type theory via a classified class of stable maps, have the required relationship for modelling dual-context type theory.

3.3.4. LEMMA. *The representable natural transformations in Lemma 3.3.3,*

$$よ(\pi, \mathsf{id}_1) : よ(\widetilde{V} \to 1) \to よ(V \to 1) \quad \textit{and} \quad よ(\flat\pi) : よ(\flat\widetilde{V}) \to よ(\flat V),$$

are related in the sense of Axiom (iv) of a fibred natural model. That is, there are natural isomorphisms

$$よ(\widetilde{V} \to 1) \circ T^{op} \cong よ(\flat\widetilde{V}) \quad \textit{and} \quad よ(V \to 1) \circ T^{op} \cong よ(\flat V).$$

Proof. Restricting the domain of $よ(\pi, \mathsf{id}_1)$ via T gives an endofunctor on $\widehat{\mathcal{C}_\flat}$ with

$$\begin{aligned}\left(よ\left(\widetilde{V} \to 1\right) \circ T^{\mathsf{op}}\right)(B) &= \left(よ\left(\widetilde{V} \to 1\right)(T(B))\right. \\ &= \mathsf{Hom}_{\mathcal{C}\downarrow\mathcal{C}_\flat}(T(B), \widetilde{V} \to 1)\end{aligned}$$

which is isomorphic to $\mathsf{Hom}_{\mathcal{C}_\flat}(B, S(\widetilde{V} \to 1))$ since S is right adjoint to T. But

$$\begin{aligned}\mathsf{Hom}_{\mathcal{C}_\flat}(B, S(\widetilde{V} \to 1)) &= \mathsf{Hom}_{\mathcal{C}_\flat}(B, \flat\widetilde{V}) \\ &= よ(\flat\widetilde{V})(B).\end{aligned}$$

□

We summarise the developments of this section with the following theorem.

3.3.5. THEOREM. *Let $\mathcal{C}$ be a small category with a specified terminal object $1_\mathcal{C}$, a stable class of maps S, and an idempotent comonad $(\flat, \varepsilon : \flat \Rightarrow id_\mathcal{C})$ that preserves the terminal object, stable maps, and pullbacks of stable maps. Suppose further that $\pi : \widetilde{V} \to V$ is a classifier for S, in the sense of Definition 2.1.7. Then there are models of dependent type theory via a category with stable maps and a classifier on the total and base categories from Remark 3.2.2, specifically:*

- *the category $\mathcal{C}\downarrow\mathcal{C}_\flat$ with stable class $S_{\mathcal{C}\downarrow\mathcal{C}_\flat}$ (Lemma 3.2.3) and classifier*

$$(\pi, \mathsf{id}_1) : (\widetilde{V}, 1, !_{\widetilde{V}} : \widetilde{V} \to 1) \to (V, 1, !_V : V \to 1)$$

  from Lemma 3.3.1
- *the category $\mathcal{C}_\flat$ with stable class $S_{\mathcal{C}_\flat}$ (Lemma 3.2.4) and classifier*

$$\flat\pi : \flat\widetilde{V} \to \flat V$$

  from Lemma 3.3.2.

These models are relative in the same sense as Axiom (iv) for a fibred natural model.

Proof. Follows from the fact that $(\mathcal{C}, 1_\mathcal{C}, \flat)$ admits a fibred natural model by Theorem 3.2.8, as well as the classifiers given by the Lemmas referenced in the theorem statement, and the relationship between them established in Lemma 3.3.4. □

3.3.6. *Terminology.* In the situation of Theorem 3.3.5, we say that $(\mathcal{C}, 1_{\mathcal{C}}, \flat)$ admits a *fibred model of dual-context type theory via categories with classified stable maps.*

We conclude by specialising Terminology 2.3.3 for vertical and horizontal display maps to the setting of a fibred natural model from a category with an idempotent comonad. We will use these notions in Chapter 4 when we develop the internal type theory of $\mathcal{C}{\downarrow}\mathcal{C}_\flat$ for $\mathcal{C}$ a presheaf category. We replace the terminology of *stable map* with *small map*, since these will be the stable maps in the presheaf setting.

3.3.7. *Remark.* We call the class of stable maps in $\mathcal{C}{\downarrow}\mathcal{C}_\flat$, $\mathsf{S}_{\mathcal{C}\downarrow\mathcal{C}_\flat}$, the *vertical small maps*, and where we have specified choices of pullbacks we call them *vertical display maps*.

3.3.8. *Remark.* A map $(f, g) : C_\bullet \to C'_\bullet$ in $\mathcal{C}{\downarrow}\mathcal{C}_\flat$ is a *horizontal small map* if it is a cartesian map with respect to $\mathsf{cod} : \mathcal{C}{\downarrow}\mathcal{C}_\flat \to \mathcal{C}_\flat$, and its projection $\mathsf{cod}(f, g) = g$ is a stable map in $\mathcal{C}_\flat$. Unfolding what this means, given a stable map $g : B' \rightarrowtriangle B$ in $\mathcal{C}_\flat$, and an object $c : C \to B$ in $\mathcal{C}{\downarrow}\mathcal{C}_\flat$, there is a pullback square

$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d, "c'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "c"] \\ B' \arrow[r, "g"'] & B, \end{tikzcd} \tag{3.4}$$

in $\mathcal{C}$, in which $f : C' \to C$ is also a stable map. Therefore, a map (f, g) is a horizontal small map in $\mathcal{C}{\downarrow}\mathcal{C}_\flat$ if and only if f and g are stable maps in $\mathcal{C}$ and they form a pullback square as in (3.4).

4 Presheaf-based models of crisp type theory

Introduction

The goal of this chapter is to extract crisp type theory as an internal language of a category. Crisp type theory is a fragment of spatial type theory, which was developed in [Shu18] with the intention of having models in any *local topos*, such as a presheaf topos on a small category with a terminal object. This is the setting of [LOPS18], where the use of crisp type theory is motivated by the specific presheaf topos of de Morgan cubical sets. Neither [Shu18] nor [LOPS18], however, specify *how* a presheaf topos admits a model of the type theory. To this end, in Section 4.1 we describe the features of a presheaf topos making it a local topos, namely a specific idempotent comonad (Observation 4.1.1), and apply Theorem 3.2.8 to show that it admits a fibred natural model of dual-context type theory. We then use the development of Section 3.3 to move from a fibred natural model to a fibred version of a category with a classified stable class of maps, since this is the style of model convenient for the applications in Chapters 5 and 6.

Sections 4.2 and 4.3 develop the theory of the comma category that features in the fibred models in Section 4.1, specifically the category-theoretic structures that correspond to type-theoretic constructions. We prove that this category is locally cartesian closed (Theorem 4.2.3) and so has left and right adjoints to pullback along any map. We then consider two special cases: adjoints to pullback along vertical small maps (from Remark 3.3.7) and along horizontal small maps (from Remark 3.3.8). These sections are analogous to Section 1 of [AGH24], which presents the properties of a presheaf category that mean dependent type theory is an internal language.

These first three sections are entirely category-theoretic. In Section 4.4, we continue to work with these semantic objects but introduce type theory-like notation to specify a dual-context type theory associated to the comma category. We show that these judgements validate the context rules of dual-context type theory. We also show that certain forms of type are supported by this language, namely standard dependent sum and dependent product types, as well as crisp product types. We do not specify all the deduction rules valid in the comma category, instead focusing on those relevant to the applications in Chapters 5 and 6. This section runs parallel to Section 3 of [AGH24], which presents the internal type theory of a presheaf category.

4.1 THE PRESHEAF-BASED MODEL

In Chapter 3, we proved that a category with an idempotent comonad admits a fibred natural model (Theorem 3.2.8). In this section, we apply this result to the intended models of crisp type theory claimed to arise from a specific idempotent comonad that exists for any presheaf category. After introducing the presheaf setting, we show that it satisfies the conditions of Theorem 3.2.8 and so admits a fibred natural model, which we give explicitly by specialising the data from Section 3.2. Finally, we use Theorem 3.3.5 to move to the setting of a fibred version of a category with a classified stable class of maps.

For the remainder of the chapter, we fix a small category $\mathbb{C}$ with a terminal object and consider the presheaf category $\widehat{\mathbb{C}}$.

4.1.1. *Observation.* There is an idempotent comonad $\flat : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ arising from the following adjunction,

$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[loop above, "\flat"] \arrow[d, bend left, "p_*"] \arrow[d, phantom, "\dashv"] \\ \mathsf{Set}, \arrow[u, bend left, "p^*"] \end{tikzcd}$$

where p_* takes a presheaf X to its set of global elements $\{x \mid x : 1 \Rightarrow X\}$, and p^* takes a set A to the constant presheaf at A, which we denote ΔA. The functor $\flat$ is defined as the composition $\flat := p^* \circ p_*$, which sends a presheaf X to the the constant presheaf on the set of global elements of X,

$$\flat X = \Delta\{x \mid x : 1 \Rightarrow X\}.$$

Explicitly, this means $\flat X(c) = \{x \mid x : 1 \Rightarrow X\}$ for all objects c in $\mathbb{C}$, and $\flat X(f) = \mathsf{id}_{\{x \mid x : 1 \Rightarrow X\}}$ for all arrows $f : d \to c$ in $\mathbb{C}$. On arrows, the functor $\flat$ sends a map $\alpha : X \Rightarrow Y$ of presheaves to a map $\flat\alpha : \flat X \Rightarrow \flat Y$, which is the natural transformation with every component given by postcomposition with α:

$$(\flat\alpha)_c(x : 1 \Rightarrow X) = (\alpha \circ x) : 1 \Rightarrow Y.$$

The counit ε of the adjunction has components $\varepsilon_X : \flat X \Rightarrow X$ that are themselves natural transformations. The component of ε_X at an object c in $\mathbb{C}$ is the function

$$(\varepsilon_X)_c : \flat X(c) \to X(c)$$

that sends a global element $x : 1 \Rightarrow X$ to $x_c(\bullet)$, where $x_c : 1(c) \to X(c)$ is a component of the global element and $x_c(\bullet)$ is its evaluation at the sole inhabitant $\bullet$ of the single element set $1(c)$.

To see that the comonad $\flat$ is idempotent, a map $\flat\flat X \Rightarrow \flat X$ is a natural transformation between constant presheaves, so is determined by a single function

$$\{y \mid y : 1 \Rightarrow \flat X\} \to \{x \mid x : 1 \Rightarrow X\}$$

that we want to show is an isomorphism. In one direction, $y : 1 \Rightarrow \flat X$ is sent to $\varepsilon_X \circ y$. In the other direction, take $x : 1 \Rightarrow X$ to the constant natural transformation $\Delta x : 1 \Rightarrow \flat X$, where every component sends the single inhabitant of 1 to x. To see this is an isomorphism, at c in $\mathbb{C}$ we have that

$$\begin{aligned}
(\varepsilon_X \circ \Delta x)_c(\bullet) &= (\varepsilon_X)_c((\Delta x)_c(\bullet)) \\
&= (\varepsilon_X)_c(x) \\
&= x_c(\bullet),
\end{aligned}$$

which is precisely the component of $x : 1 \Rightarrow X$ at c. The other way around, at an object c in $\mathbb{C}$ we have

$$\Delta(\varepsilon_X \circ y)_c(\bullet) = \varepsilon_X \circ y$$

so we need to show that $\varepsilon_X \circ y = y_c(\bullet)$ as maps $1 \Rightarrow X$. A component of $\varepsilon_X \circ y$ at an object d in $\mathbb{C}$ evaluates as

$$\begin{aligned}
(\varepsilon_X \circ y)_d(\bullet) &= (\varepsilon_X)_d(y_d(\bullet)) \\
&= \varepsilon_X(y_d(\bullet))
\end{aligned}$$

since $y_c(\bullet) = y_d(\bullet)$, because y is a map between constant presheaves and so is determined by a single function. But

$$(\varepsilon_X)_d(y_c(\bullet)) = (y_c(\bullet))_d(\bullet),$$

which is the component of $y_c(\bullet) : 1 \Rightarrow X$ at d.

4.1.2. *Remark.* The intended models of Shulman's spatial type theory [Shu18], of which crisp type theory is a fragment, are *local toposes*. Since $\mathbb{C}$ was assumed to have a terminal object, $\widehat{\mathbb{C}}$ is a local topos [Joh02, Section C3.6], the defining condition of which is that the global sections/constant presheaf adjunction $p^* \dashv p_*$ has a further right adjoint $p^!$. What is more, $\widehat{\mathbb{C}}$ is a *cohesive* topos [Law07] since p^* has a left adjoint $p_!$ that preserves finite products. This amounts to an adjoint quadruple

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[d, shift right=9, "p_! \dashv"] \arrow[d, shift right=3, leftarrow, "p^* \dashv"] \arrow[d, shift left=3, "p_* \dashv"] \arrow[d, shift left=9, leftarrow, "p^!"] \\
\mathsf{Set},
\end{tikzcd}$$

which induces the following adjoint triple of endofunctors on $\widehat{\mathbb{C}}$:

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[d, shift right=6, leftarrow, "ʃ \dashv"] \arrow[d, "\flat \dashv"] \arrow[d, shift left=6, leftarrow, "\sharp"] \\
\widehat{\mathbb{C}}
\end{tikzcd}$$

We use the fact that $\flat$ is a right adjoint in the next result, which is a consequence of Theorem 3.2.8.

4.1.3. THEOREM. *The category of presheaves $\widehat{\mathbb{C}}$ on any small category $\mathbb{C}$ admits a fibred natural model (Definition 2.3.1) via the idempotent comonad $(\flat, \varepsilon)$ in Observation 4.1.1. Specifically, there is a fibred natural model of dual-context type theory over the codomain functor cod : $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat \to \widehat{\mathbb{C}}_\flat$, recalling from Notation 3.2.1 that*

- *$\widehat{\mathbb{C}}_\flat$ is the full subcategory of $\widehat{\mathbb{C}}$ spanned by objects M with $\varepsilon_M = \mathrm{id}_M$*
- *$\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ is the category with objects*

$$\begin{tikzcd} X \arrow[d, "\chi"] \\ M \end{tikzcd}$$

where X is an object in $\widehat{\mathbb{C}}$ and M is an object in the subcategory $\widehat{\mathbb{C}}_\flat$, and arrows $(f, g) : \psi \to \chi$ are commutative squares

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\psi"'] & X \arrow[d, "\chi"] \\ N \arrow[r, "g"'] & M \end{tikzcd}$$

where f is an arrow in $\widehat{\mathbb{C}}$ and g is an arrow in $\widehat{\mathbb{C}}_\flat$.

The remaining data of the model comes from specialising Remark 3.2.2 as follows,

$$\begin{tikzcd} \widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat \arrow[d, "cod"', bend right=0] \arrow[d, phantom, shift left=2, "\dashv"] \arrow[d, shift left=8, "S"] \arrow[d, phantom, shift left=6, "\dashv"] \\ \widehat{\mathbb{C}}_\flat \arrow[u, shift left=4, "T"] \end{tikzcd} \tag{4.1}$$

where

- *T is the terminal functor, given on an object N and an arrow $u : N \to M$ in $\widehat{\mathbb{C}}_\flat$ by*

$$T(N) = (N, N, \mathrm{id}_N)$$
$$T(u) = (u, u),$$

- *S is the comprehension functor, given on objects and arrows by*

$$S(X, M, \chi : X \to M) = \flat X$$
$$S(f, g) = \flat f.$$

Proof. We verify that this setting satisfies the hypotheses of Theorem 3.2.8. Certainly, $\widehat{\mathbb{C}}$ has a terminal object, as well as a stable class of maps given by the class of small maps $\mathcal{S}$ defined in Example 2.1.5. The idempotent comonad is a right adjoint (see Remark 4.1.2) and so it preserves limits, namely the terminal object and all pullbacks. Suppose $p : A \twoheadrightarrow X$ is a small map in $\widehat{\mathbb{C}}$, then $\flat p : \flat A \Rightarrow \flat X$ is the natural transformation with every component given by postcomposing $a : 1 \Rightarrow A$ in $\flat A(c)$ (for any c in $\mathbb{C}$) with p. Since a is clearly a small map, the composition $p \circ a$ is also small. □

The next proposition simplifies the presheaf-based fibred model by showing that the base category is equivalent to the category of sets.

4.1.4. PROPOSITION. *Let* $(\flat, \varepsilon)$ *be the idempotent comonad on* $\widehat{\mathbb{C}}$ *in Observation 4.1.1. There is an equivalence of categories*

$$\widehat{\mathbb{C}}_\flat \simeq \mathsf{Set},$$

where $\widehat{\mathbb{C}}_\flat$ *is the full subcategory of* $\widehat{\mathbb{C}}$ *spanned by objects* M *with* $\varepsilon_M = \mathsf{id}_M$.

Proof. Associated with the adjunction $p^* \dashv p_*$ from Observation 4.1.1 and the induced comonad $\flat$ is the comparison functor

$$\Delta : \mathsf{Set} \to \widehat{\mathbb{C}}_\flat$$

that sends a set A to the constant presheaf on A and a function $f : A \to B$ to the natural transformation $\Delta f : \Delta A \Rightarrow \Delta B$, in which each component is given by $f : A \to B$. This functor is essentially surjective on objects, since for any X in $\widehat{\mathbb{C}}_\flat$, $X \cong \flat X$ and $\flat X = \Delta\{x \mid x : 1 \to X\}$, so X is mapped to from the set $\{x \mid x : 1 \to X\}$. To see that it is also fully faithful and thus defines an equivalence of categories, consider the induced function on hom-sets

$$\Delta_{\mathsf{Hom}} : \mathsf{Hom}_{\mathsf{Set}}(A, B) \to \mathsf{Hom}_{\widehat{\mathbb{C}}_\flat}(\Delta A, \Delta B).$$

Since Δ_{Hom} sends an element $f : A \to B$ of $\mathsf{Hom}_{\mathsf{Set}}(A, B)$ to the natural transformation with every component f, it is clear to see that $\Delta_{\mathsf{Hom}}(f) = \Delta_{\mathsf{Hom}}(g)$ implies $f = g$. Furthermore, an element $\alpha : \Delta A \Rightarrow \Delta B$ of $\mathsf{Hom}_{\widehat{\mathbb{C}}_\flat}(\Delta A, \Delta B)$ is just a natural transformation with every component given by a single function $f : A \to B$, so $\alpha = \Delta_{\mathsf{Hom}}(f)$. □

Fibred category with stable maps and a classifier from $(\widehat{\mathbb{C}}, \flat)$

There is a classifier for small maps in the category $\widehat{\mathbb{C}}$, namely the Hofmann-Streicher universe $\pi : E \to U$ from Example 2.1.11, so we can apply the development of Section 3.3 to move from the setting of a fibred natural model to that of a fibred version of a category with stable maps and a classifier. Before specifiying the classifiers inherited by the total category $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ and base category $\widehat{\mathbb{C}}_\flat$, we state the relevant stable classes of maps in these categories.

4.1.5. *Observation.* The comma category $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ and the subcategory $\widehat{\mathbb{C}}_\flat$ each inherit a stable class of maps from the class of small maps $\mathcal{S}$ in $\widehat{\mathbb{C}}$ as follows.

(i) By Lemma 3.2.3, there is a stable class of maps in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ of the form

$$(p, \mathsf{id}_M) : (A, M, \alpha : A \to M) \to (X, M, \chi : X \to M)$$

where $p : A \rightarrowtriangle X$ is in $\mathcal{S}$. Such a map in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ corresponds to a commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "p"'] \arrow[d, "\alpha"] \\ X \arrow[d, "\chi"'] & M \arrow[dl, equal] \\ M. & \end{tikzcd}$$

(ii) By Lemma 3.2.4, there is a stable class of maps in the subcategory $\widehat{\mathbb{C}}_\flat$ given by small maps $u : N \twoheadrightarrow M$ such that $\flat u = u$.

4.1.6. *Observation.* The stable classes of maps in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ and $\widehat{\mathbb{C}}_\flat$ each inherit a classifier from the fact that $\pi : E \to U$ classifies small maps in $\widehat{\mathbb{C}}$.

(i) By Proposition 3.3.1, the stable class of maps in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ (Observation 4.1.5) are classified by the map

$$\begin{tikzcd} & E \arrow[dl, "\pi"'] \arrow[d, "\mathcal{E}"] \\ U \arrow[d, "\mathcal{U}"'] & 1, \arrow[dl, equal] \\ 1 & \end{tikzcd} \tag{4.2}$$

which we denote $(\pi, \mathsf{id}_1) : \mathcal{E} \to \mathcal{U}$. The objects $\mathcal{U} : U \to 1$ and $\mathcal{E} : E \to 1$ are just the unique maps in $\widehat{\mathbb{C}}$ into the terminal object.

(ii) By Proposition 3.3.2, the stable class of maps in $\widehat{\mathbb{C}}_\flat$ (Observation 4.1.5) are classified by the map

$$\flat\pi : \flat E \to \flat U. \tag{4.3}$$

4.1.7. *Remark.* The choice of pullback squares of $\pi : E \to U$, detailed in Remark 2.1.12, induce a choice of pullback squares of $\pi : \mathcal{E} \to \mathcal{U}$.

Towards working with $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ as a category that has a dual-context type theory as an internal language, we specialise the vertical and horizontal small maps of Remarks 3.3.7 and 3.3.8 to the presheaf-based model.

4.1.8. *Remark.* Let $(f, g) : \psi \to \chi$ be a map in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$, as in

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\psi"'] & X \arrow[d, "\chi"] \\ N \arrow[r, "g"'] & M. \end{tikzcd}$$

Then (f, g) is

(a) a *vertical small map* just if it is in the stable class of maps in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ from Observation 4.1.5. In this case, we use the naming conventions

$$\begin{tikzcd} & A \arrow[dl, "p"'] \arrow[d, "\alpha"] \\ X \arrow[d, "\chi"'] & M \arrow[dl, equal] \\ M & \end{tikzcd}$$

and we refer to this class of maps as $\mathcal{V}$.

(b) a *horizontal small map* if it is a cartesian map with respect to $\mathrm{cod} : \widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat \to \widehat{\mathbb{C}}_\flat$, and its projection $\mathrm{cod}(f, g) = g$ is a small map in $\widehat{\mathbb{C}}_\flat$. From Remark 3.3.8, this is the case just when f

and g are small maps in $\widehat{\mathbb{C}}$ forming a pullback square. In this case, we standardly write

$$\begin{tikzcd} Y \arrow[r, "q", -{Triangle[open]}] \arrow[d, "\psi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "\chi"] \\ N \arrow[r, "r"', -{Triangle[open]}] & M. \end{tikzcd}$$

Recall that in a presheaf category $\widehat{\mathbb{C}}$, we have three equivalent perspectives: a small map $p : A \rightarrowtriangle X$, a map $\sigma : X \to U$, and a display map $p_\sigma : X.\sigma \rightarrowtriangle X$ (Remark 2.1.12). The next remarks are analogues of this for vertical and horizontal small maps in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$.

4.1.9. *Remark.* There are three equivalent perspectives for vertical small maps in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$:

(i) a vertical small map $(p, \mathsf{id}_M) : \alpha \to \chi$,

$$\begin{tikzcd} & A \arrow[dl, "p"', -{Triangle[open]}] \arrow[d, "\alpha"] \\ X \arrow[d, "\chi"'] & M \\ M \arrow[ur, equal] & \end{tikzcd}$$

(ii) a map $(\sigma, !_M) : \chi \to \mathcal{U}$,

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\chi"'] & U \arrow[d, "\mathcal{U}"] \\ M \arrow[r] & 1 \end{tikzcd}$$

(iii) a vertical display map $(p_\sigma, \mathsf{id}_M) : \chi.\sigma \to \chi$,

$$\begin{tikzcd} & X.\sigma \arrow[dl, "p_\sigma"', -{Triangle[open]}] \arrow[d, "\chi.\sigma"] \\ X \arrow[d, "\chi"'] & M \\ M \arrow[ur, equal] & \end{tikzcd}$$

obtained by pullback of $\pi : \mathcal{E} \to \mathcal{U}$ along $(\sigma, !_M) : \chi \to \mathcal{U}$, using Remark 4.1.7.

4.1.10. *Remark.* There are three equivalent perspectives for horizontal small maps in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$:

(i) a horizontal small map $(r, q) : \alpha \to \chi$,

$$\begin{tikzcd} A \arrow[r, "r", -{Triangle[open]}] \arrow[d, "\alpha"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "\chi"] \\ N \arrow[r, "q"', -{Triangle[open]}] & M \end{tikzcd} \tag{4.4}$$

(ii) a map $\sigma : M \to \flat U$ in $\widehat{\mathbb{C}}_\flat$,

$$M \xrightarrow{\sigma} \flat U,$$

which itself can be viewed in three ways by Remark 2.1.12, as in the diagram

$$\begin{tikzcd}
N \arrow[dr, dashed, "\cong"] \arrow[drr, bend left] \arrow[ddr, bend right, "q"'] & & \\
& M.\sigma \arrow[r] \arrow[d, "q_\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & \flat E \arrow[d, "\flat\pi"] \\
& M \arrow[r, "\sigma"'] & \flat U,
\end{tikzcd}$$

(iii) a horizontal display map $q_\sigma : M.\sigma \to M$ in $\widehat{\mathbb{C}}_\flat$

$$\begin{tikzcd}
M.\sigma \arrow[r] \arrow[d, "q_\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & \flat E \arrow[d, "\flat\mathsf{ty}"] \\
M \arrow[r, "\sigma"'] & \flat U
\end{tikzcd}$$

with cod-cartesian lift to χ in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ given by $(r_\sigma, q_\sigma) : \chi{*}\sigma \to \chi$:

$$\begin{tikzcd}
X{*}\sigma \arrow[r, "r_\sigma"] \arrow[d, "\chi{*}\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\chi"] \\
M.\sigma \arrow[r, "q_\sigma"'] & M,
\end{tikzcd} \tag{4.5}$$

where r_σ is the chosen cartesian lift. The relationship between the diagrams in (4.4) and (4.5) is given by the diagrams

$$\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr, bend left] \arrow[dd, bend right, "\alpha"'] & & \\
& X{*}\sigma \arrow[r, "r_\sigma"] \arrow[d, "\chi{*}\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\chi"] \\
N \cong & M.\sigma \arrow[r, "q_\sigma"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
X{*}\sigma \arrow[dr, dashed] \arrow[drr, bend left] \arrow[dd, bend right, "\chi{*}\sigma"'] & & \\
& A \arrow[r, "r"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\chi"] \\
M.\sigma \cong & N \arrow[r, "q"'] & M.
\end{tikzcd}$$

where the dashed arrows induced by the pullback property mean $A \cong X{*}\sigma$.

4.2 The comma category of presheaves

Recall that the goal of this chapter is to present crisp type theory as an internal language of a category, namely the comma category $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ introduced in the previous section. Before defining this internal language in Section 4.4, we lay the groundwork by establishing the properties of the comma category necessary for validating type-theoretic rules. Foremost is that $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ is locally cartesian closed and so has left and right adjoints to pullback along any map. These adjoints will then be studied in more detail in Section 4.3. We also show that $\widehat{\mathbb{C}}$ is a reflective subcategory of $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$, which will allow us to relate the standard internal type theory of a presheaf category to that of $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$.

4.2.1. *Remark.* We will prove that $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ is locally cartesian closed by showing that it is equivalent to another locally cartesian closed category, namely a (co)presheaf category. We use the observation that, since $\widehat{\mathbb{C}}_\flat$ is equivalent to Set (Proposition 4.1.4), an object $\chi : X \to M$ in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ is just a cocone under the diagram $X : \mathbb{C}^{\mathrm{op}} \to \mathrm{Set}$ with apex M in Set, as in

$$\chi : X \Rightarrow \Delta M,$$

where ΔM the constant presheaf at M. Then the following lemma uses the fact that a cocone in a category $\mathbb{D}$ can be thought of as adjoining a terminal object to $\mathbb{D}$.

4.2.2. LEMMA. *Let $\mathbb{D}$ be a small category and $[\mathbb{D}, \mathsf{Set}]$ denote a functor category (that is, objects are functors $\mathbb{D} \to \mathsf{Set}$ and arrows are natural transformations between them). Let $[\mathbb{D}, \mathsf{Set}]{\downarrow}\mathsf{Set}$ denote the comma category with:*

- *objects $\iota : F \Rightarrow \Delta A$ that are natural transformations from $F : \mathbb{D} \to \mathsf{Set}$ to the constant functor $\Delta A : \mathbb{D} \to \mathsf{Set}$ on a set A*
- *arrows given by a pair of natural transformations $(\alpha : F \Rightarrow G, \Delta g : \Delta A \Rightarrow \Delta B)$, where Δg is the constant natural transformation with every component the function $g : A \to B$, such that the following commutes:*

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\iota"'] & G \arrow[d, Rightarrow, "\kappa"] \\ \Delta A \arrow[r, Rightarrow, "\Delta g"'] & \Delta B. \end{tikzcd} \tag{4.6}$$

Let $\mathbb{D}_T$ be the category $\mathbb{D}$ with a terminal object T adjoined, meaning that in addition to the new object T, for every object c in $\mathbb{D}$ there is a map $!_c : c \to T$ in $\mathbb{D}_T$ such that for any map $f : c \to d$ in $\mathbb{D}$, there is a commutative triangle

$$\begin{tikzcd} c \arrow[r, "!_c"] \arrow[d, "f"'] & T \\ d \arrow[ur, "!_d"'] & \end{tikzcd} \tag{4.7}$$

in $\mathbb{D}_T$.

Then there is an equivalence of categories

$$[\mathbb{D}, \mathsf{Set}]{\downarrow}\mathsf{Set} \simeq [\mathbb{D}_T, \mathsf{Set}],$$

where $[\mathbb{D}_T, \mathsf{Set}]$ denotes a functor category.

Proof. The desired equivalence is given by a functor

$$\overline{(\_)} : [\mathbb{D}, \mathsf{Set}]{\downarrow}\mathsf{Set} \to [\mathbb{D}_T, \mathsf{Set}] \tag{4.8}$$

that takes $\iota : F \Rightarrow \Delta A$ to the functor $\overline{F} : \mathbb{D}_T \to \mathsf{Set}$ given on objects by

$$\overline{F}(c) := \begin{cases} F(c) & \text{for } c \in \mathbb{D} \\ A & \text{for } c = T, \end{cases} \tag{4.9}$$

and on an arrow $f : c \to d$ in $\mathbb{D}_T$ by

$$\overline{F}(f) := \begin{cases} F(f) & \text{for } f : c \to d \in \mathbb{D} \\ \iota_c & \text{for } f = !_c : c \to T \in \mathbb{D}_T. \end{cases} \tag{4.10}$$

This defines a functor because, in addition to the functoriality of F, applying $\overline{F}$ to any diagram of the form in (4.7) as in

$$\begin{tikzcd} \overline{F}(c) \arrow[r, "\overline{F}(!_c)"] \arrow[d, "\overline{F}(f)"'] & \overline{F}(T) \\ \overline{F}(d) \arrow[ur, "\overline{F}(!_d)"'] & \end{tikzcd}$$

evaluates to

$$\begin{tikzcd} F(c) \arrow[r, "\iota_c"] \arrow[d, "F(f)"'] & A \\ F(d), \arrow[ur, "\iota_d"'] & \end{tikzcd}$$

which commutes by the naturality of ι.

To finish specifying the functor in (4.8), an arrow

$$(\alpha, \Delta g) : \iota \to \kappa$$

in $[\mathbb{D}, \mathrm{Set}]\downarrow\mathrm{Set}$, as in (4.6), gets sent to a natural transformation $\overline{\alpha} : \overline{F} \Rightarrow \overline{G}$ with component at c in $\mathbb{D}_T$ given by

$$\overline{\alpha}_c := \begin{cases} \alpha_c, & \text{for } c \in \mathbb{D} \\ g, & \text{for } c = T. \end{cases}$$

This is natural by the naturality of α and the fact that, for an arrow $!_c : c \to T$ in $\mathbb{D}_T$, the following square commutes:

$$\begin{tikzcd} \overline{F}(c) \arrow[r, "\overline{\alpha}_c"] \arrow[d, "F(!_c)"'] & \overline{G}(c) \arrow[d, "G(!_c)"] \\ \overline{F}(T) \arrow[r, "\overline{\alpha}_T"'] & \overline{G}(T). \end{tikzcd} \tag{4.11}$$

This is because (4.11) evaluates to

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "\iota_c"'] & G(c) \arrow[d, "\kappa_c"] \\ A \arrow[r, "g"'] & B, \end{tikzcd}$$

which is the diagram in (4.6) at c in $\mathbb{D}$.

To see that the functor we have just defined is essentially surjective, let $Z : \mathbb{D}_T \to \mathrm{Set}$ and let $i : \mathbb{D} \to \mathbb{D}_T$ be the subcategory inclusion. Then there is an object

$$\zeta : Z \circ i \Rightarrow \Delta Z(T)$$

in $[\mathbb{D}, \mathrm{Set}]\downarrow\mathrm{Set}$, with component at c in $\mathbb{D}$

$$\zeta_c : Z(c) \to Z(T)$$

defined to be

$$\zeta_c := Z(!_c).$$

The object ζ is natural because of the functoriality of Z and the fact that T is terminal in $\mathbb{D}_T$, meaning we have the following naturality square for any $f : c \to d$ in $\mathbb{D}$:

$$\begin{tikzcd} Z(c) \arrow[r, "\zeta_c"] \arrow[d, "Z(f)"'] & Z(T) \arrow[d, equal] \\ Z(d) \arrow[r, "\zeta_d"'] & Z(T). \end{tikzcd}$$

It is straightforward to see that $\overline{Z \circ i} = Z$ using the definition of the functor $\overline{(\_)}$ in (4.9) and (4.10):

$$\begin{aligned} &\overline{Z \circ i}(c) = Z \circ i(c) = Z(c) \text{ for } c \text{ in } \mathbb{D} \\ &\overline{Z \circ i}(T) = Z(T) \\ &\overline{Z \circ i}(f) = Z(f) \text{ for } f : c \to d \text{ in } \mathbb{D} \\ &\overline{Z \circ i}(!_c) = \zeta_c = Z(!_c) \text{ for } !_c : c \to T \text{ in } \mathbb{D}_T. \end{aligned}$$

The functor is also fully faithful, since the induced function on hom-sets

$$\mathsf{Hom}_{[\mathbb{D},\mathsf{Set}]\downarrow\mathsf{Set}}(\iota, \kappa) \to \mathsf{Hom}_{[\mathbb{D}_T,\mathsf{Set}]}(\overline{F}, \overline{G})$$

has an inverse, as follows. Given a natural transformation $\beta : \overline{F} \Rightarrow \overline{G}$, restricting β by the subcategory inclusion $i : \mathbb{D} \to \mathbb{D}_T$ gives a map

$$\beta \circ i : F \Rightarrow G.$$

Since $\overline{F}(T) = A$ and $\overline{G}(T) = B$, the component of β at T is a function $\beta_T : A \to B$. Therefore we have a map $(\beta \circ i, \Delta\beta_T)$ fitting into the following diagram

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\beta \circ i"] \arrow[d, Rightarrow, "\iota"'] & G \arrow[d, Rightarrow, "\kappa"] \\ \Delta A \arrow[r, Rightarrow, "\Delta\beta_T"'] & \Delta B \end{tikzcd} \tag{4.12}$$

since (4.12) on components at c in $\mathbb{D}$

$$\begin{tikzcd} F(c) \arrow[r, "(\beta \circ i)_c"] \arrow[d, "\iota_c"'] & G(c) \arrow[d, "\kappa_c"] \\ A \arrow[r, "\beta_T"'] & B \end{tikzcd}$$

commutes as it equal to the following naturality square for β and the map $!_c : c \to T$:

$$\begin{tikzcd} \overline{F}(c) \arrow[r, "\beta_c"] \arrow[d, "\overline{F}(!_c)"'] & \overline{G}(c) \arrow[d, "\overline{G}(!_c)"] \\ \overline{F}(T) \arrow[r, "\beta_T"'] & \overline{G}(T). \end{tikzcd}$$

□

4.2.3. THEOREM. *The category $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ is locally cartesian closed.*

Proof. Letting $\mathbb{D}$ in Lemma 4.2.2 be $\mathbb{C}^{\mathrm{op}}$ and using the fact that $\widehat{\mathbb{C}}_\flat$ is equivalent to Set (Proposition 4.1.4) we have the following equivalence of categories:

$$\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat \simeq [(\mathbb{C}^{\mathrm{op}})_T, \mathsf{Set}].$$

Then since (co)presheaf categories are locally cartesian closed, so is $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. □

Next we establish the relationship between $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. This will be used in Section 4.4 to specify when the internal type theory of $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ can be simplified to that of $\widehat{\mathbb{C}}$ (Proposition 4.4.19).

4.2.4. PROPOSITION. *The category $\widehat{\mathbb{C}}$ is a reflective full subcategory of $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$.*

Proof. There is an adjunction given by

$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[r, bend right, "!_{(-)}"'] \arrow[r, phantom, "\perp"] & \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat \arrow[l, bend right, "\mathsf{dom}"'] \end{tikzcd}$$

where dom is the domain functor, and $!_{(-)}$ sends a presheaf X to the unique map into the terminal object in $\widehat{\mathbb{C}}$,

$$\begin{tikzcd} X \arrow[d, "!_X"'] \\ 1, \end{tikzcd}$$

and an arrow $f : X \to Y$ to the arrow $(f, \mathsf{id}_1) : !_X \to !_Y$, as in

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "!_X"'] & Y \arrow[d, "!_Y"] \\ 1 \arrow[r, equal] & 1. \end{tikzcd}$$

This is an adjunction since for any Y in $\widehat{\mathbb{C}}$ and $\chi : X \to M$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, there is an isomorphism

$$\mathsf{Hom}_{\widehat{\mathbb{C}}}(\mathsf{dom}(\chi), Y) \cong \mathsf{Hom}_{\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat}(\chi, !_Y)$$

natural in both Y and χ, taking a map $f : X \to Y$ to the map

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi"'] & Y \arrow[d, "!_Y"] \\ M \arrow[r, "!_M"'] & 1 \end{tikzcd}$$

in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, which commutes by the universal property of the terminal object. For any object X in $\widehat{\mathbb{C}}$, the component of the counit

$$\eta_X : \mathsf{dom} \circ !_{(-)}(X) \to X$$

is an isomorphism. □

4.3 Dependent sums and products in the comma category

Having established that $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ is locally cartesian closed (Theorem 4.2.3), we now consider left and right adjoints to pullback along maps from our special classes: vertical small maps and horizontal small maps (Remark 4.1.8). Foreshadowing their role in the next section, adjoints to pullback along vertical small maps will correspond to standard dependent sum and dependent product types, while adjoints to pullback along horizontal small maps will correspond to *crisp* dependent sum and dependent product types. This section is still purely category-theoretic, to separate out the category-theoretic and type-theoretic developments.

Small maps in $\widehat{\mathbb{C}}$ and vertical small maps in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$

In setting up the internal type theory of a presheaf category $\widehat{\mathbb{C}}$, Awodey et al. [AGH24] specify the closure properties of small maps that mean the language supports certain forms of type. We wish to apply that development to the internal type theory of $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. In particular, the vertical small maps of a fibred model of dual-context type theory exist within a single fibre—thought of as fixing the first context zone—and thus operations on them look like operations in standard dependent type theory. We formalise this as an equivalence of categories between (slices of) small maps and vertical small maps, with a view to using the closure properties of small maps for vertical small maps.

4.3.1. *Notation.*

(i) For an object X in $\widehat{\mathbb{C}}$, we write $\mathcal{S}_X$ for the full subcategory of $\widehat{\mathbb{C}}/X$ spanned by small maps. It has objects $q : B \twoheadrightarrow X$ and arrows $k : q' \to q$, as in

$$\begin{tikzcd} B' \arrow[rr, "k"] \arrow[dr, "q'"'] & & B \arrow[dl, "q"] \\ & X. & \end{tikzcd} \tag{4.13}$$

(ii) For an object $\chi : X \to M$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, we write $\mathcal{V}_\chi$ for the full subcategory of $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi$ spanned by vertical small maps. It has objects $(p, \mathrm{id}_M) : \alpha \to \chi$, as in

$$\begin{tikzcd} & A \arrow[dl, "p"'] \arrow[d, "\alpha"] \\ X \arrow[d, "\chi"'] & M \arrow[dl, equal] \\ M, & \end{tikzcd} \tag{4.14}$$

and arrows $(j, \mathrm{id}_M) : (p', \mathrm{id}_M) \to (p, \mathrm{id}_M)$, as in

$$\begin{tikzcd} A' \arrow[rr, "j"] \arrow[d, "\alpha'"'] \arrow[dr, "p'"] & & A \arrow[dl, "p"'] \arrow[d, "\alpha"] \\ M \arrow[r, equal] \arrow[dr, equal] & X \arrow[d, "\chi"] & M \arrow[l, equal] \arrow[dl, equal] \\ & M. & \end{tikzcd} \tag{4.15}$$

4.3.2. PROPOSITION. *Let X be an object in $\widehat{\mathbb{C}}$, $\chi : X \to M$ be an object in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, and $\mathcal{S}_X$ and $\mathcal{V}_\chi$ be the categories in Notation 4.3.1. There is an isomorphism of categories*

$$\mathcal{S}_X \cong \mathcal{V}_\chi.$$

Proof. For an object $(p, \mathsf{id}_M) : \alpha \to \chi$ in $\mathcal{V}_\chi$, as in (4.14), the map $\alpha : A \to M$ is determined by $\chi \circ p$. It is then clear to see there is an isomorphism given by functors

$$\begin{tikzcd} \mathcal{V}_\chi \arrow[r, shift left, "\mathsf{dom}"] & \mathcal{S}_X \arrow[l, shift left, "\chi\circ-"] \end{tikzcd}$$

where

- the domain functor dom is defined for an object (p, id_M), as in (4.14), and an arrow $(j, \mathsf{id}_M) : (p', \mathsf{id}_M) \to (p, \mathsf{id}_M)$, as in (4.15), by

$$\mathsf{dom}(p, \mathsf{id}_M) := p$$
$$\mathsf{dom}(j, \mathsf{id}_M) := j$$

- the functor $\chi\circ-$ is defined for an object $q : B \rightarrowtriangle X$ by

$$(\chi\circ-)(q) := (q, \mathsf{id}_M) : \chi\circ q \to \chi,$$

as in

$$\begin{tikzcd} & B \arrow[dl, "q"'] \arrow[d, "\chi\circ q"] \\ X \arrow[d, "\chi"'] & M \arrow[dl, equal] \\ M, & \end{tikzcd}$$

and for an arrow $k : q' \to q$, as in (4.13), by

$$(\chi\circ-)(k) := (k, \mathsf{id}_M) : (q', \mathsf{id}_M) \to (q, \mathsf{id}_M),$$

as in

$$\begin{tikzcd} B' \arrow[rr, "k"] \arrow[dr, "q'"] \arrow[dd, "\chi\circ q'"'] & & B \arrow[dl, "q"'] \arrow[dd, "\chi\circ q"] \\ & X \arrow[dd, "\chi"] & \\ M \arrow[rr, equal] \arrow[dr, equal] & & M \arrow[dl, equal] \\ & M. & \end{tikzcd}$$

It is straightforward to see that the composites $(\chi\circ-)\circ\mathsf{dom}$ and $\mathsf{dom}\circ(\chi\circ-)$ are equal to the identity functors on $\mathcal{V}_\chi$ and $\mathcal{S}_X$ respectively. □

Dependent sums and products of vertical and horizontal small maps

Since $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ is locally cartesian closed (Theorem 4.2.3), for any map $(f, g) : \psi \to \chi$, as in

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\psi"'] & X \arrow[d, "\chi"] \\ N \arrow[r, "g"'] & M, \end{tikzcd} \tag{4.16}$$

the pullback functor

$$(f, g)^* : \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi \to \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\psi$$

has both a left and a right adjoint, denoted

$$\begin{tikzcd} \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi \arrow[rr, "(f,g)^*" description] \arrow[rr, phantom, bend left=25, "\perp"] \arrow[rr, phantom, bend right=25, "\perp"] & & \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\psi \arrow[ll, bend right=50, "\Sigma_{(f,g)}"'] \arrow[ll, bend left=50, "\Pi_{(f,g)}"] \end{tikzcd} \tag{4.17}$$

We are interested in two cases: when (f, g) is a vertical small map and when (f, g) is a horizontal small map. Before considering each case, we recall a remark about small maps in $\widehat{\mathbb{C}}$ from [AGH24, p.6].

4.3.3. *Remark.* If $p : A \rightarrowtriangle X$ is a small map in $\widehat{\mathbb{C}}$, then the left and right adjoints to pullback along p in $\widehat{\mathbb{C}}$ restrict to small maps. That is, there are serially-commuting diagrams

$$\begin{tikzcd} \mathcal{S}_A \arrow[r, bend left, "\Sigma_p"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{S}_X \arrow[l, bend left, "p^*"] \arrow[d] \\ \widehat{\mathbb{C}}/A \arrow[r, bend left, "\Sigma_p"] \arrow[r, phantom, "\perp"] & \widehat{\mathbb{C}}/X \arrow[l, bend left, "p^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}_X \arrow[r, bend left, "p^*"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{S}_A \arrow[l, bend left, "\Pi_p"] \arrow[d] \\ \widehat{\mathbb{C}}/X \arrow[r, bend left, "p^*"] \arrow[r, phantom, "\perp"] & \widehat{\mathbb{C}}/A. \arrow[l, bend left, "\Pi_p"] \end{tikzcd}$$

Similarly, left and right adjoints to pullback along (f, g) in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ restrict to vertical small maps when (f, g) is either a vertical small map or a horizontal small map. Specifically, let $\alpha : A \to M$ and $\chi : X \to M$ be objects in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ and let $(p, \mathrm{id}_M) : \alpha \to \chi$ be a vertical small map in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, as in Remark 4.1.9. Then there are serially-commuting diagrams

$$\begin{tikzcd} \mathcal{V}_\alpha \arrow[r, bend left, "\Sigma_{(p,\mathrm{id}_M)}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{V}_\chi \arrow[l, bend left, "(p,\mathrm{id}_M)^*"] \arrow[d] \\ \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\alpha \arrow[r, bend left, "\Sigma_{(p,\mathrm{id}_M)}"] \arrow[r, phantom, "\perp"] & \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi \arrow[l, bend left, "(p,\mathrm{id}_M)^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{V}_\chi \arrow[r, bend left, "(p,\mathrm{id}_M)^*"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{V}_\alpha \arrow[l, bend left, "\Pi_{(p,\mathrm{id}_M)}"] \arrow[d] \\ \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi \arrow[r, bend left, "(p,\mathrm{id}_M)^*"] \arrow[r, phantom, "\perp"] & \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\alpha. \arrow[l, bend left, "\Pi_{(p,\mathrm{id}_M)}"] \end{tikzcd} \tag{4.18}$$

Now let $\alpha : A \to N$ and $\chi : X \to M$ be objects in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ and $(q, r) : \alpha \to \chi$ be a horizontal display map between them, as in Remark 4.1.10. Then there are serially-commuting diagrams

$$
\begin{tikzcd}
\mathcal{V}_\alpha \arrow[r, bend left, "\Sigma_{(r,q)}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{V}_\chi \arrow[l, bend left, "(r,q)^*"] \arrow[d] \\
\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\alpha \arrow[r, bend left, "\Sigma_{(r,q)}"] \arrow[r, phantom, "\perp"] & \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi \arrow[l, bend left, "(r,q)^*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}_\chi \arrow[r, bend left, "(r,q)^*"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{V}_\alpha \arrow[l, bend left, "\Pi_{(r,q)}"] \arrow[d] \\
\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi \arrow[r, bend left, "(r,q)^*"] \arrow[r, phantom, "\perp"] & \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\alpha. \arrow[l, bend left, "\Pi_{(r,q)}"]
\end{tikzcd}
\tag{4.19}
$$

Given a fixed $\chi : X \to M$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, we will use the above remark and the isomorphism between $\mathcal{S}_X$ and $\mathcal{V}_\chi$ in the following remarks about dependent products of vertical and horizontal small maps in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. We note that there are analogous results for dependent sums of vertical and horizontal small maps, but focus on dependent products as these correspond to the form of type used in applications in Chapter 6.

4.3.4. *Remark* (Dependent products of vertical display maps). Let $\chi : X \to M$ be an object in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ and suppose there are maps $\sigma : \chi \to U$ and $\tau : \chi.\sigma \to U$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, that is,

$$
\begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[d, "\chi"'] & U \arrow[d] \\
M \arrow[r] & 1
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X.\sigma \arrow[r, "\tau"] \arrow[d, "\chi.\sigma"'] & U \arrow[d] \\
M \arrow[r] & 1.
\end{tikzcd}
\tag{4.20}
$$

Let $p_\sigma : \chi.\sigma \to \chi$ and $p_\tau : \chi.\sigma.\tau \to \chi.\sigma$ be their associated vertical display maps, as in

$$
\begin{tikzcd}
& X.\sigma \arrow[dl, "p_\sigma"'] \arrow[d, "\chi.\sigma"] \\
X \arrow[d, "\chi"'] & M \arrow[dl, equal] \\
M &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& X.\sigma.\tau \arrow[dl, "p_\tau"'] \arrow[d, "\chi.\sigma.\tau"] \\
X.\sigma \arrow[d, "\chi.\sigma"'] & M. \arrow[dl, equal] \\
M &
\end{tikzcd}
$$

Since p_σ and p_τ are vertical small maps in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, we obtain a vertical small map $\Pi_{p_\sigma}(p_\tau)$ using the adjunction in (4.18): specifically, by letting α be $\chi.\sigma : X.\sigma \to M$ and p be $p_\sigma : \chi.\sigma \to \chi$, then applying the functor Π_{p_σ} to p_τ. The action of the adjoint is then reflected into an operation on classifying maps as follows. Since Π_{p_σ} is a horizontal small map, it is classified; let the classifier be called $\Pi_\sigma(\tau)$ and observe that it gives rise to a pullback diagram

$$
\begin{tikzcd}
& X.\Pi_\sigma(\tau) \arrow[rr] \arrow[dl] \arrow[dd, "\Pi_\sigma(\tau)"] & & E \arrow[dl] \arrow[dd] \\
X \arrow[rr, crossing over] \arrow[dd] & & U & \\
& M \arrow[rr] \arrow[dl, equal] & & 1 \arrow[dl, equal] \\
M \arrow[rr] & & 1 \arrow[from=uu, crossing over] &
\end{tikzcd}
\tag{4.21}
$$

in which the $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$-morphism on the left hand side is isomorphic to $\Pi_{p_\sigma}(p_\tau)$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi$ and hence inherits its universal property.

Sections of this vertical display map can be characterised as follows, using the result for $\widehat{\mathbb{C}}$ in [AGH24, Proposition 1.5].

4.3.5. PROPOSITION. *Let $\sigma : \chi \to U$ and $\tau : \chi.\sigma \to U$ be arrows in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ as in (4.20). Then the following data are in bijective correspondence:*

(i) sections $(b, \mathrm{id}_M) : \chi.\sigma \to \chi.\sigma.\tau$ over $\chi.\sigma$,

(ii) sections $(b^\#, \mathrm{id}_M) : \chi \to \chi.\Pi_\sigma(\tau)$ over χ.

Proof. By Proposition 4.3.2, we have

$$\begin{tikzcd} \mathcal{S}_X \arrow[r, bend left, "p_\sigma^*"] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{S}_A \arrow[l, bend left, "\Pi_{p_\sigma}"] \arrow[d, phantom, "\cong" rotate=90] \\ \mathcal{V}_\chi \arrow[r, bend left, "p_\sigma^*"] \arrow[r, phantom, "\perp"] & \mathcal{V}_\alpha \arrow[l, bend left, "\Pi_{p_\sigma}"] \end{tikzcd}$$

Therefore, the result follows from the same reasoning as [AGH24, Proposition 1.5]. $\square$

4.3.6. *Remark* (Dependent products and sums of horizontal display maps). Given an object $\chi : X \to M$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ and a map $\sigma : M \to \flat U$ in $\widehat{\mathbb{C}}_\flat$, let $(r, q) : \chi{*}\sigma \to \chi$ be the associated horizontal display map

$$\begin{tikzcd} X{*}\sigma \arrow[r, "r"] \arrow[d, "\chi{*}\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\chi"] \\ M.\sigma \arrow[r, "q"'] & M, \end{tikzcd}$$

as in Remark 4.1.10. Furthermore, given $\tau : \chi{*}\sigma \to \mathcal{U}$, as in

$$\begin{tikzcd} X{*}\sigma \arrow[r, "\tau"] \arrow[d, "\chi{*}\sigma"'] & U \arrow[d] \\ M.\sigma \arrow[r] & 1, \end{tikzcd}$$

let $p : \chi * \sigma.\tau \to \chi{*}\sigma$ be the associated vertical display map

$$\begin{tikzcd} & X{*}\sigma.\tau \arrow[dl, "p"'] \arrow[d, "\chi{*}\sigma.\tau"] \\ X{*}\sigma \arrow[d, "\chi{*}\sigma"'] & M.\sigma \arrow[dl, equal] \\ M.\sigma, & \end{tikzcd} \tag{4.22}$$

as in Remark 4.1.9.

Therefore, we can specialise the adjunction in Diagrams 4.19 to get a right adjoint to pullback along (r, q):

$$\begin{tikzcd} \mathcal{V}_\chi \arrow[r, bend left, "(r,q)^*"] \arrow[r, phantom, "\perp"] & \mathcal{V}_{\chi * \flat\sigma} \arrow[l, bend left, "\Pi_{(r,q)}"] \end{tikzcd} \tag{4.23}$$

Applying $\Pi_{(r,q)}$ to the vertical display map $p : \chi{*}\flat\sigma.\tau \to \chi{*}\flat\sigma$ from the diagram in 4.22 gives an object $\Pi_{(r,q)}p$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi$ that must also be a vertical display map in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. Let $\Pi_{\flat\sigma}\tau$ be the classifying map of the stable map $\Pi_{(r,q)}p$. This gives rise to the following pullback diagram

$$
\begin{tikzcd}[row sep=small, column sep=small]
& X.\Pi_{\flat\sigma}(\tau) \arrow[rr] \arrow[dl, "p_{\Pi_{\flat\sigma}(\tau)}"'] \arrow[dd, "\chi.\Pi_{\flat\sigma}(\tau)" near end] & & E \arrow[dl] \arrow[dd] \\
X \arrow[rr, "\Pi_{\flat\sigma}(\tau)" near start, crossing over] \arrow[dd] & & U & \\
& M \arrow[rr] \arrow[dl, equal] & & 1 \arrow[dl, equal] \\
M \arrow[rr] & & 1 \arrow[from=uu, crossing over] &
\end{tikzcd}
\tag{4.24}
$$

in which the vertical display map on the left hand side is isomorphic to $\Pi_{(r,q)}p$ when viewed as objects in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi$, and hence inherits its universal property. In particular, take the counit of the adjunction in 4.23 and consider its component at $p : \chi{*}\flat\sigma.\tau \to \chi{*}\flat\sigma$,

$$\varepsilon_p : (r,q)^* \circ \Pi_{(r,q)}p \to p.$$

Since $\Pi_{(r,q)}p$ is isomorphic to $p_{\Pi_{\flat\sigma}(\tau)}$, the object $(r,q)^* \circ \Pi_{(r,q)}p$ is isomorphic to the left hand map in the pullback of $p_{\Pi_{\flat\sigma}(\tau)}$ along (r,q), which we call u:

$$
\begin{tikzcd}[row sep=small, column sep=small]
& X{*}\flat\sigma.\Pi_{\flat\sigma}(\tau) \arrow[rr] \arrow[dl, "u"'] \arrow[dd] & & X.\Pi_{\flat\sigma}(\tau) \arrow[dl, "p_{\Pi_{\flat\sigma}(\tau)}"] \arrow[dd] \\
X{*}\flat\sigma \arrow[rr, "r" near end, crossing over] \arrow[dd, "\chi{*}\flat\sigma"'] & & X & \\
& M.\flat\sigma \arrow[rr] \arrow[dl, equal] & & M \arrow[dl, equal] \\
M.\flat\sigma \arrow[rr, "q"'] & & M \arrow[from=uu, crossing over] &
\end{tikzcd}
\tag{4.25}
$$

Therefore, the counit induces a map in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi{*}\flat\sigma$ that we call $\varepsilon : u \to p$:

$$
\begin{tikzcd}
X{*}\flat\sigma.\Pi_{\flat\sigma}\tau \arrow[rr, "\varepsilon"] \arrow[dr, "u"] \arrow[dd, "\chi{*}\flat\sigma.\Pi_{\flat\sigma}\tau"'] & & X{*}\flat\sigma.\tau \arrow[dl, "p"'] \arrow[dd, "\chi{*}\flat\sigma.\tau"] \\
& X{*}\flat\sigma \arrow[dd, "\chi{*}\flat\sigma" near start] & \\
M.\flat\sigma \arrow[rr, equal] \arrow[dr, equal] & & M.\flat\sigma \arrow[dl, equal] \\
& M.\flat\sigma &
\end{tikzcd}
\tag{4.26}
$$

We now use this remark to characterise sections of the display map $p_{\Pi_{\flat\sigma}(\tau)}$ in (4.24), analogously to Proposition 1.5 [AGH24] for a presheaf category.

4.3.7. PROPOSITION. *Let $\sigma : \mathrm{id}_M \to \mathcal{U}$ and $\tau : \chi{*}\flat\sigma \to \mathcal{U}$, as in*

$$
\begin{tikzcd}
M \arrow[r, "\sigma"] \arrow[d, equal] & U \arrow[d] \\
M \arrow[r] & 1
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X{*}\flat\sigma \arrow[r, "\tau"] \arrow[d, "\chi{*}\flat\sigma"] & U \arrow[d] \\
M.\flat\sigma \arrow[r] & 1,
\end{tikzcd}
$$

where $\chi{*}\flat\sigma$ is obtained by pullback of the object $\chi : X \to M$ along the horizontal display map $\flat\sigma : M.\flat\sigma \to M$, as detailed in Remark 4.1.10. Then the following data are in bijective correspondence:

(i) sections $s : \chi{*}\flat\sigma \to \chi{*}\flat\sigma.\tau$ over $\chi{*}\flat\sigma$:

$$\begin{tikzcd}
X{*}\flat\sigma \arrow[rr, "s"] \arrow[dr, equal] \arrow[dd, "\chi{*}\flat\sigma"'] & & X{*}\flat\sigma.\tau \arrow[rr] \arrow[dl, "p"] \arrow[dd] & & E \arrow[dl] \arrow[dd] \\
& X{*}\flat\sigma \arrow[rr, "\tau"] \arrow[dd] & & U \arrow[dd] & \\
M.\flat\sigma \arrow[rr, equal] \arrow[dr, equal] & & M.\flat\sigma \arrow[rr] \arrow[dl, equal] & & 1 \arrow[dl, equal] \\
& M.\flat\sigma \arrow[rr] & & 1 &
\end{tikzcd} \tag{4.27}$$

(ii) sections $s^\sharp : \chi \to \chi.\Pi_{\flat\sigma}(\tau)$ over χ:

$$\begin{tikzcd}
X \arrow[rr, "s^\sharp"] \arrow[dr, equal] \arrow[dd, "\chi"'] & & X.\Pi_{\flat\sigma}(\tau) \arrow[rr] \arrow[dl, "p_{\Pi_{\flat\sigma}(\tau)}"'] \arrow[dd] & & E \arrow[dl] \arrow[dd] \\
& X \arrow[rr, "\Pi_{\flat\sigma}(\tau)"] \arrow[dd] & & U \arrow[dd] & \\
M \arrow[rr, equal] \arrow[dr, equal] & & M \arrow[rr] \arrow[dl, equal] & & 1 \arrow[dl, equal] \\
& M \arrow[rr] & & 1 &
\end{tikzcd} \tag{4.28}$$

Proof. For the first direction, assume we have $s : \chi{*}\sigma \to \chi{*}\flat\sigma.\tau$ as in (4.27). We can view s as an arrow $s : \mathsf{id}_{\chi{*}\flat\sigma} \to p$ in the slice $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi{*}\flat\sigma$. Then we want an arrow $s^\sharp : \mathsf{id}_\chi \to p_{\Pi_{\flat\sigma}(\tau)}$ in the slice $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi$. Let

$$\eta_{\mathsf{id}_\chi} : \mathsf{id}_\chi \to \Pi_{(q,r)} \circ (r,q)^*(\mathsf{id}_\chi)$$

be the component at id_χ of the unit of the adjunction in (4.23). Noting that $(r,q)^*(\mathsf{id}_\chi) = \mathsf{id}_{\chi{*}\flat\sigma}$, let $s^\sharp$ be the following composition

$$\mathsf{id}_\chi \xrightarrow{\eta_{\mathsf{id}_\chi}} \Pi_{(q,r)}(\mathsf{id}_{\chi{*}\flat\sigma}) \xrightarrow{\Pi_{(q,r)}s} \Pi_{(q,r)}p \xrightarrow{\cong} p_{\Pi_{\flat\sigma}(\tau)}$$

where the final isomorphism comes from 4.24, where $p_{\Pi_{\flat\sigma}(\tau)}$ is the display map associated with the small map $\Pi_{(q,r)}p$, in accordance with Remark 4.1.10.

For the second direction, assume we have $s^\sharp$ as in (4.28), viewed as an arrow $s^\sharp : \mathsf{id}_\chi \to p_{\Pi_{\flat\sigma}(\tau)}$ in the slice $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi$. Applying the pullback functor $(r,q)^*$ to $s^\sharp$ gives a map

$$(r,q)^*(s^\sharp) : \mathsf{id}_{\chi{*}\flat\sigma} \to u$$

where $u = (r,q)^*(p_{\Pi_{\flat\sigma}(\tau)})$ as in the diagram in (4.25).

$$\begin{tikzcd}
X{*}\flat\sigma \arrow[rr, "(r{,}q)^*(s^\sharp)"] \arrow[dr, equal] \arrow[dd, "\chi{*}\flat\sigma"'] & & X{*}\flat\sigma.\Pi_{\flat\sigma}(\tau) \arrow[dl, "u"] \arrow[dd, "\chi{*}\flat\sigma.\Pi_{\flat\sigma}(\tau)"] \\
& X{*}\flat\sigma \arrow[dd, "\chi{*}\flat\sigma"] & \\
M.\flat\sigma \arrow[rr, equal] \arrow[dr, equal] & & M.\flat\sigma \arrow[dl, equal] \\
& M.\flat\sigma &
\end{tikzcd} \tag{4.29}$$

The desired map comes from composing this with ε in the diagram in (4.26):

$$s = \varepsilon \circ (r, q)^*(s^\sharp)$$

$\square$

A similar result can be obtained for dependent sums but is not included here since they are not required for the applications in Chapter 6.

4.4 THE TYPE THEORY OF THE COMMA CATEGORY

We now specify how $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ determines a dual-context type theory via the classifier $(\pi, \mathrm{id}_1) : \mathcal{E} \to \mathcal{U}$ (Equation 4.2). We begin by introducing some terminology.

4.4.1. DEFINITION (the core of the language $\mathcal{T}_{\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat}$).

- A *context* is an object

$$\begin{tikzcd}
\Theta \arrow[d, "\chi"] \\
\Delta
\end{tikzcd}$$

  of $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. We call Θ the *total context* and Δ the *crisp* or *modal* part of the context. In this case, we write $\Theta \vdash_\Delta$.
- For a context χ, a *type in context* χ is a morphism $(\sigma, !_\Delta) : \chi \to \mathcal{U}$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, given diagrammatically as

$$\begin{tikzcd}
\Theta \arrow[r, "\sigma"] \arrow[d, "\chi"'] & U \arrow[d, "\mathcal{U}"] \\
\Delta \arrow[r] & 1.
\end{tikzcd} \tag{4.30}$$

  In this case, we write $\Theta \vdash_\Delta \sigma : \mathcal{U}$.
- Given two types σ and τ in context χ, we say that σ and τ are *judgementally equal* if they are equal as morphisms from χ to $\mathcal{U}$. In this case, we write $\Theta \vdash_\Delta \sigma = \tau : \mathcal{U}$.
- For a context χ and a type σ in context χ, an *element of* σ in context χ is a map $(s, !_\Delta) : \chi \to \mathcal{E}$ such that

$$\begin{tikzcd}
 & & E \arrow[dl, "\pi"'] \arrow[dd, "\varepsilon"] \\
\Theta \arrow[r, "\sigma"'] \arrow[urr, bend left, "s"] \arrow[dd, "\chi"'] & U \arrow[dd] & \\
 & & 1 \\
\Delta \arrow[r] \arrow[urr, bend left] & 1 \arrow[ur, equal] &
\end{tikzcd} \tag{4.31}$$

  commutes. In this case, we write $\Theta \vdash_\Delta s : \sigma$.
- For two terms s_1 and s_2 of type σ in context χ, we say that s_1 and s_2 are *judgementally equal* elements of σ if they are equal as morphisms from χ to $\mathcal{E}$. In this case, we write $\chi \vdash s_1 = s_2 : \sigma$.

4.4.2. *Notation.* A map in the comma category $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ comprises a map in $\widehat{\mathbb{C}}$ and a map in $\widehat{\mathbb{C}}_\flat$. When one of these maps is trivial, we will omit it from the name. For example, the map $(\sigma, !_\Delta) : \chi \to \mathcal{U}$ in (4.30) will be written $\sigma : \chi \to \mathcal{U}$.

4.4.3. *Remark.* The *empty context* is the terminal object $\mathsf{id}_1 : 1 \to 1$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. A type in the empty context, known as a *closed type*, is a map $\sigma : \mathsf{id}_1 \to \mathcal{U}$, as in

$$\begin{tikzcd} 1 \arrow[r, "\sigma"] \arrow[d, equal] & U \arrow[d, "\mathcal{U}"] \\ 1 \arrow[r, equal] & 1, \end{tikzcd}$$

for which we write $1 \vdash_1 \sigma : \mathcal{U}$.

4.4.4. *Remark.* The *purely crisp contexts* are the objects in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ that are identity maps in $\widehat{\mathbb{C}}_\flat$, that is, χ is of the form $\mathsf{id}_\Delta : \Delta \to \Delta$ for Δ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. In this case, we say that the *non-crisp part of the context is empty*, and we write $\Delta \vdash_\Delta$.

4.4.5. *Remark.* The notation for a context (Definition 4.4.1), an empty context (Remark 4.4.3), and a purely crisp context (Remark 4.4.4) just introduced relate to the notation for contexts in crisp type theory in [Shu18] as follows. For an object $\chi : \Theta \to \Delta$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$,

$$\Theta \vdash_\Delta \quad \leftrightsquigarrow \quad \Delta \mid \Gamma \ \mathsf{ctx}$$

The domain Θ corresponds to the total context $\Delta \mid \Gamma$ in crisp type theory, while the codomain Δ corresponds to the crisp component. Informally, using the intuition of the syntactic category, χ may be thought of as projection of the first part of the context. For a purely crisp context and the empty context:

$$\begin{aligned} \Delta \vdash_\Delta \quad &\leftrightsquigarrow \quad \Delta \mid \cdot \ \mathsf{ctx} \\ 1 \vdash_1 \quad &\leftrightsquigarrow \quad \cdot \mid \cdot \ \mathsf{ctx} \end{aligned}$$

4.4.6. DEFINITION (standard context extension). For a context $\chi : \Theta \to \Delta$ and a type σ in context χ, there is a new context obtained by the specified pullback from Remark 4.1.7:

$$\begin{tikzcd}[row sep=small, column sep=small] & \Theta.\sigma \arrow[rr] \arrow[dl, "p_\sigma"'] \arrow[dd, "\chi.\sigma" near end] & & E \arrow[dl] \arrow[dd] \\ \Theta \arrow[rr, "\sigma" near start, crossing over] \arrow[dd, "\chi"'] & & U & \\ & \Delta \arrow[rr] \arrow[dl, equal] & & 1 \arrow[dl, equal] \\ \Delta \arrow[rr] & & 1 \arrow[from=uu, crossing over] & \end{tikzcd} \tag{4.32}$$

We call this operation *standard* or *non-crisp context extension* of χ by σ. We denote the new context by $\chi.\sigma : \Theta.\sigma \to \Delta$ and write $\Theta.\sigma \vdash_\Delta$.

4.4.7. *Remark.* An analogous remark to [AGH24, Remark 3.2] applies in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. Elements of a type σ in context χ are in bijective correspondence with sections of its display map $p_\sigma : \chi.\sigma \to \chi$. Another way to view this is that an element s of σ in context χ determines the diagram

$$\begin{tikzcd}
\Theta \arrow[rrrd, bend left, "s"] \arrow[rrd, dashed, "\langle \mathrm{id}_\Theta, s\rangle"] \arrow[dd] \arrow[ddr, equal] & & & & \\
& & \Theta.\sigma \arrow[rr] \arrow[dl, "p_\sigma"'] \arrow[dd] & & E \arrow[dl] \arrow[dd] \\
\Delta \arrow[rrd, dashed] \arrow[ddr, equal] & \Theta \arrow[rr, "\sigma"] \arrow[dd] & & U \arrow[dd] & \\
& & \Delta \arrow[rr] & & 1 \\
& \Delta \arrow[rr] \arrow[ur, equal] & & 1 \arrow[ur, equal] &
\end{tikzcd} \tag{4.33}$$

where the dashed arrows form the $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$-morphism $(\langle \mathrm{id}_\Theta, s\rangle, \mathrm{id}_\Delta)$.

If σ is a type in a purely crisp context, then there is a second kind of context extension by σ.

4.4.8. DEFINITION (crisp context extension). For a context $\mathrm{id}_\Delta : \Delta \to \Delta$ and a type σ in context id_Δ, that is, a map $\sigma : \mathrm{id}_\Delta \to \mathcal{U}$, there is a new context obtained by the operation of *crisp context extension*.

(i) Apply $S : \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat \to \widehat{\mathbb{C}}_\flat$ from 4.1 to σ,

$$\Delta \xrightarrow{\flat\sigma} \flat U,$$

noting that $\flat\Delta = \Delta$.

(ii) Take the pullback in $\widehat{\mathbb{C}}_\flat$ of the classifier along $\flat\sigma$:

$$\begin{tikzcd}
\Delta.\flat\sigma \arrow[r, "w_{\flat\sigma}"] \arrow[d, "q_{\flat\sigma}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \flat E \arrow[d, "\flat\pi"] \\
\Delta \arrow[r, "\flat\sigma"'] & \flat U.
\end{tikzcd} \tag{4.34}$$

(iii) An object in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ is then obtained by taking the cod-cartesian lift of $q_{\flat\sigma}$ to any context with crisp component Δ, that is, of the form $\chi : \Theta \to \Delta$.

(a) In general, we have

$$\begin{tikzcd}
\Theta{*}\sigma \arrow[r, "r_{\flat\sigma}"] \arrow[d, "\chi{*}\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta \arrow[d, "\chi"] \\
\Delta.\flat\sigma \arrow[r, "q_{\flat\sigma}"'] & \Delta,
\end{tikzcd} \tag{4.35}$$

obtaining the new context $\chi{*}\sigma : \Theta{*}\sigma \to \Delta.\flat\sigma$, for which we write $\Theta{*}\sigma \vdash_{\Delta.\sigma}$.

(b) If $\chi : \Theta \to \Delta$ is $\mathrm{id}_\Delta : \Delta \to \Delta$, then

$$\begin{tikzcd}
\Delta.\flat\sigma \arrow[r, "q_{\flat\sigma}"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta \arrow[d, equal] \\
\Delta.\flat\sigma \arrow[r, "q_{\flat\sigma}"'] & \Delta,
\end{tikzcd}$$

obtaining the new context $\mathrm{id}_{\Delta.\flat\sigma} : \Delta.\flat\sigma \to \Delta.\flat\sigma$, for which we write $\Delta.\sigma \vdash_{\Delta.\sigma}$.

4.4.9. *Remark.* We summarise these two kinds of context extensions—and the two cases of crisp context extension—by noting that $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ validates the following rules:

$$\frac{\Theta \vdash_\Delta \sigma : \mathcal{U}}{\Theta.\sigma \vdash_\Delta} \qquad \frac{\Delta \vdash_\Delta \sigma : \mathcal{U}}{\Delta.\sigma \vdash_{\Delta.\sigma}} \qquad \frac{\Delta \vdash_\Delta \sigma : \mathcal{U} \qquad \Theta \vdash_\Delta}{\Theta{*}\sigma \vdash_{\Delta.\sigma}}$$

We introduce the following terminology for maps in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$.

4.4.10. DEFINITION (context morphism). Let $\chi : \Theta \to \Delta$ and $\chi' : \Theta' \to \Delta'$ be contexts. A *context morphism* from χ' to χ is a map (f, g)

$$\begin{tikzcd} \Theta' \arrow[r, "f"] \arrow[d, "\chi'"'] & \Theta \arrow[d, "\chi"] \\ \Delta' \arrow[r, "g"'] & \Delta \end{tikzcd}$$

in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$.

Context morphisms act on types and elements via the following operation of *substitution*.

4.4.11. *Notation* (substitution). Let $\chi : \Theta \to \Delta$ and $\chi' : \Theta' \to \Delta'$ be contexts and fix a context morphism $(f, g) : \chi' \to \chi$. For a type σ in context χ, we define a type $\sigma(f, g)$ in context χ', obtained by *substitution* of (f, g) in σ, by letting

$$\sigma(f, g) := (\sigma, !_\Delta)\circ(f, g),$$

as in the following diagram:

$$\begin{tikzcd} \Theta' \arrow[r, "f"] \arrow[d, "\chi'"'] & \Theta \arrow[r, "\sigma"] \arrow[d, "\chi"] & U \arrow[d] \\ \Delta' \arrow[r, "g"'] & \Delta \arrow[r, "!_\Delta"'] & 1. \end{tikzcd}$$

We have the following diagram of associated context extensions, in which the left cube is also a pullback:

$$\begin{tikzcd} & \Theta'.\sigma(f,g) \arrow[rr] \arrow[dl] \arrow[dd] & & \Theta.\sigma \arrow[rr] \arrow[dl] \arrow[dd] & & E \arrow[dl] \arrow[dd] \\ \Theta' \arrow[rr, "f"] \arrow[dd, "\chi'"'] & & \Theta \arrow[rr, "\sigma"] \arrow[dd] & & U \arrow[dd] & \\ & \Delta' \arrow[rr] \arrow[dl, equal] & & \Delta \arrow[rr] \arrow[dl, equal] & & 1 \arrow[dl, equal] \\ \Delta' \arrow[rr, "g"'] & & \Delta \arrow[rr] & & 1. & \end{tikzcd}$$

If s is a term of type σ in context χ, we define the *substitution* of (f, g) in s to be the element $s(f, g)$ of type $\sigma(f, g)$ in context χ' defined by letting

$$s(f, g) := (s, !_\Delta)\circ(f, g),$$

as in the diagram

$$\begin{tikzcd}
 & & & E \arrow[dl, "\pi"'] \arrow[dd, "\varepsilon"] \\
\Theta' \arrow[r, "f"] \arrow[dd, "\chi'"'] & \Theta \arrow[r, "\sigma"] \arrow[dd, "\chi"] \arrow[urr, bend left, "s"] & U \arrow[dd] & \\
 & & & 1 \\
\Delta' \arrow[r, "g"'] & \Delta \arrow[r] \arrow[urr, bend left] & 1. \arrow[ur, equal] &
\end{tikzcd}$$

Substitution satisfies the expected rules, such as

$$\sigma(f,g)(f',g') = \sigma(f \circ f', g \circ f')$$

for $(f',g') : \chi'' \to \chi'$.

In the particular case of substitution of a context morphism induced by a section of a display map, as in Remark 4.4.7, we simplify the notation as follows. Given a type τ in context $\chi.\sigma$, we write the substitution of $(\langle \mathsf{id}_\Theta, s\rangle, \mathsf{id}_\Delta)$ in τ as $\tau(s)$, instead of $\tau(\langle \mathsf{id}_\Theta, s\rangle, \mathsf{id}_\Delta)$. And if t is a term of type τ in context $\chi.\sigma$, we write $t(s)$ rather than $t(\langle \mathsf{id}_\Theta, s\rangle, \mathsf{id}_\Delta)$. This corresponds to the diagram

$$\begin{tikzcd}
 & & & E \arrow[dl, "\pi"'] \arrow[dd, "\varepsilon"] \\
\Theta \arrow[r, "(\mathsf{id}_\Theta{,}s)"] \arrow[dd, "\chi"'] & \Theta.\sigma \arrow[r, "\tau"] \arrow[dd, "\chi.\sigma"'] \arrow[urr, bend left, "t"] & U \arrow[dd] & \\
 & & & 1 \\
\Delta \arrow[r, equal] & \Delta \arrow[r] \arrow[urr, bend left] & 1. \arrow[ur, equal] &
\end{tikzcd}$$

With this notation in hand, we can see that the following substitution rule is validated:

$$\frac{\Theta \vdash_\Delta s : \sigma \qquad \Theta.\sigma \vdash_\Delta t : \tau}{\Theta \vdash_\Delta t(s) : \tau(s)}$$

4.4.12. *Notation* (crisp substitution). Let $\mathsf{id}_\Delta : \Delta \to \Delta$ be a context. A term s of type σ in context id_Δ corresponds to a section $s : \Delta \to \Delta.\sigma$ in $\widehat{\mathbb{C}}_\flat$ of the associated display map $q_\sigma : \Delta.\sigma \to \Delta$. Now suppose that we have the following judgement in the internal language,

$$\Delta.\sigma.\rho \vdash_{\Delta.\sigma} t : \tau$$

which corresponds to the diagram

$$\begin{tikzcd}
 & & E \arrow[dl, "\pi"'] \arrow[dd, "\varepsilon"] \\
\Delta.\sigma.\rho \arrow[r, "\tau"'] \arrow[dd, "\mathsf{id}_\Delta * \sigma.\rho"'] \arrow[urr, bend left, "t"] & U \arrow[dd] & \\
 & & 1 \\
\Delta.\sigma \arrow[r] \arrow[urr, bend left] & 1. \arrow[ur, equal] &
\end{tikzcd}$$

Then we can perform the operation of *crisp substitution* by taking the pullback of the extended context $\mathsf{id}_\Delta{*}\sigma.\rho : \Delta.\sigma.\rho \to \Delta.\sigma$ along the section $s : \Delta \to \Delta.\sigma$, as in the following:

$$\begin{tikzcd}
 & & & E \arrow[dl, "\pi"'] \arrow[dd, "\varepsilon"] \\
\Delta.\rho(s) \arrow[r, "s'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.\sigma.\rho \arrow[r, "\tau"] \arrow[d, "\mathsf{id}_\Delta{*}\sigma.\rho"'] \arrow[urr, bend left, "t"] & U \arrow[d] & \\
 & & & 1 \\
\Delta \arrow[r, "s"'] & \Delta.\sigma \arrow[r] \arrow[urr, bend left] & 1. \arrow[ur, equal] &
\end{tikzcd}$$

Naming the object formed by pullback $\Delta.\rho(s)$ and the map $s' : \Delta.\rho(s) \to \Delta.\sigma.\rho$, we can see that the following crisp substitution rule is validated:

$$\frac{\Delta \vdash_\Delta s : \sigma \qquad \Delta.\sigma.\rho \vdash_{\Delta.\sigma} t : \tau}{\Delta.\rho(s) \vdash_\Delta t(s') : \tau(s')}$$

4.4.13. *Notation* (standard weakening). Let σ and τ each be types in context $\chi : \Theta \to \Delta$. We can perform substitution for σ along the display map $p_\tau : \chi.\tau \twoheadrightarrow \chi$ as follows

$$\begin{tikzcd}
 & \Theta.\tau.\sigma \arrow[rr] \arrow[dl] \arrow[dd] & & \Theta.\sigma \arrow[rr] \arrow[dl] \arrow[dd] & & E \arrow[dl] \arrow[dd] \\
\Theta.\tau \arrow[rr, "p_\tau"] \arrow[dd, "\chi.\tau"'] & & \Theta \arrow[rr, "\sigma"] \arrow[dd] & & U \arrow[dd] & \\
 & \Delta \arrow[rr, equal] \arrow[dl, equal] & & \Delta \arrow[rr] \arrow[dl, equal] & & 1 \arrow[dl, equal] \\
\Delta \arrow[rr, equal] & & \Delta \arrow[rr] & & 1. &
\end{tikzcd}$$

This means the following rule is validated, in which the context of σ is weakened:

$$\frac{\Theta \vdash_\Delta \sigma : \mathcal{U} \qquad \Theta \vdash_\Delta \tau : \mathcal{U}}{\Theta.\tau \vdash_\Delta \sigma(p_\tau) : \mathcal{U}}$$

4.4.14. *Notation* (weakening of the crisp context). The crisp part of the context may also be weakened as follows. Let σ be a type in context $\mathsf{id}_\Delta : \Delta \to \Delta$ and τ be a type in context $\chi : \Theta \to \Delta$. The context of τ may be weakened crisply by σ by composition with the context morphism in 4.35, that is, the cod-cartesian lift of the display map $q_{\flat\sigma} : \Delta.\flat\sigma \to \Delta$ in $\widehat{\mathbb{C}}_\flat$ to χ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$:

$$\begin{tikzcd}
\Theta{*}\sigma \arrow[r, "r_{\flat\sigma}", -Triangle] \arrow[d, "\chi{*}\sigma"'] & \Theta \arrow[r, "\tau"] \arrow[d, "\chi"] & U \arrow[d] \\
\Delta.\flat\sigma \arrow[r, "q_{\flat\sigma}"', -Triangle] & \Delta \arrow[r] & 1.
\end{tikzcd} \tag{4.36}$$

This corresponds to the following rule being valid in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$:

$$\frac{\Delta \vdash_\Delta \sigma : \mathcal{U} \qquad \Theta \vdash_\Delta \tau : \mathcal{U}}{\Theta{*}\sigma \vdash_{\Delta.\sigma} \tau : \mathcal{U}}$$

4.4.15. *Terminology.* We refer to the system of contexts, context morphisms, types, elements and the context extension operations given above as the *internal language of* $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ and we denote it by $\mathcal{T}_{\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat}$.

$$\frac{\Theta \vdash_\Delta \sigma : \mathcal{U}}{\Theta.\sigma \vdash_\Delta}\ \textsc{ctx/ext/regular} \qquad \frac{\Delta \vdash_\Delta \sigma : \mathcal{U}}{\Delta.\sigma \vdash_{\Delta.\sigma}}\ \textsc{ctx/ext/crisp} \qquad \frac{\Delta \vdash_\Delta \sigma : \mathcal{U} \qquad \Theta \vdash_\Delta}{\Theta{*}\sigma \vdash_{\Delta.\sigma}}\ \textsc{ctx/wkn/crisp}$$

$$\frac{\Theta \vdash_\Delta s : \sigma \qquad \Theta.\sigma \vdash_\Delta t : \tau}{\Theta \vdash_\Delta t(s) : \tau(s)}\ \textsc{ctx/sub/regular} \qquad \frac{\Delta \vdash_\Delta s : \sigma \qquad \Delta.\sigma.\rho \vdash_{\Delta.\sigma} t : \tau}{\Delta.\rho(s) \vdash_\Delta t(s') : \tau(s')}\ \textsc{ctx/sub/crisp}$$

$$\frac{\Theta \vdash_\Delta \sigma : \mathcal{U} \qquad \Theta \vdash_\Delta \tau : \mathcal{U}}{\Theta.\tau \vdash_\Delta \sigma(p_\tau) : \mathcal{U}}\ \textsc{ctx/wkn/regular} \qquad \frac{\Delta \vdash_\Delta \sigma : \mathcal{U} \qquad \Theta \vdash_\Delta \tau : \mathcal{U}}{\Theta{*}\sigma \vdash_{\Delta.\sigma} \tau : \mathcal{U}}\ \textsc{ctx/wkn/crisp}$$

$$\frac{\Delta \vdash_\Delta \sigma : \mathcal{U} \qquad \Theta{*}\sigma \vdash_{\Delta,x::\sigma} \tau(x) : \mathcal{U}}{\Theta \vdash_\Delta \Pi_{x::\sigma}\tau(x) : \mathcal{U}}\ \textsc{pi/crisp/form} \qquad \frac{\Theta{*}\sigma \vdash_{\Delta,x::\sigma} t : \tau(x)}{\Theta \vdash_\Delta \lambda(x :: \sigma).t : \Pi_{x::\sigma}\tau(x)}\ \textsc{pi/crisp/intro}$$

$$\frac{\Theta{*}\sigma \vdash_{\Delta,x::\sigma} f : \Pi_{x::\sigma}\tau(x) \qquad \Delta \vdash_\Delta s : \sigma}{\Theta \vdash_\Delta fs : \tau[s/x]}\ \textsc{pi/crisp/elim}$$

FIGURE 4.1: Rules of $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$

Crisp Π*-types*

With the notation introduced in Section 4.4 specifying the internal language $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$, we can show that the rules for crisp Π-types given in [LOPS18, p.8] are valid in $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$, in the sense that given the semantic objects corresponding to the premise of a rule, we can form the semantic object in the conclusion of the rule. In this way, the rules in 4.4.16 concern semantic objects and are not to be considered the generating rules of a dependent type theory.

4.4.16. PROPOSITION (Rules for crisp Π-types). *The following rules are valid in* $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$.

- *Formation: If* $\Delta \vdash_\Delta \sigma : \mathcal{U}$ *and* $\Theta{*}\sigma \vdash_{\Delta,x::\sigma} \tau(x) : \mathcal{U}$ *then* $\Theta \vdash_\Delta \Pi_{x::\sigma}\tau(x) : \mathcal{U}$
- *Introduction: If* $\Theta{*}\sigma \vdash_{\Delta,x::\sigma} t : \tau(x)$ *then* $\Theta \vdash_\Delta \lambda(x :: \sigma).t : \Pi_{x::\sigma}\tau(x)$
- *Elimination: If* $\Theta \vdash_\Delta f : \Pi_{x::\sigma}\tau(x)$ *and* $\Delta \vdash_\Delta s : \sigma$ *then* $\Theta \vdash_\Delta fs : \tau[s/x]$

Proof. The judgements in these rules correspond to diagrams in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ using the notation specifying the language $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$. Their validity then follows from results established for $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ in Section 4.2. Specifically, the formation rule is just Remark 4.3.6, and the introduction and elimination rules are Proposition 4.3.7. □

The internal type theory $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$

We summarise the development in this chapter with the following theorem.

4.4.17. THEOREM. *The rules in Figure 4.1 are valid in* $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$.

4.4.18. *Remark.* All the rules of crisp type theory (Figure 1.1) are contained in the rules of the internal language $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ (Figure 4.1). Therefore, crisp type theory is a subtheory of $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$.

We conclude with two remarks for working practically with the internal type theory of the comma category. These will be used in Section 6.3 when we relate the diagrammatic and type-theoretic constructions of the universal uniform fibration.

4.4.19. *Remark.* When the crisp part of the context is empty, as in judgements with the context

$$\Theta \vdash_1$$

corresponding to an object

$$\begin{array}{c}\Theta\\ \big\downarrow\\ 1\end{array}$$

in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$, the type theory $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ simplifies to the standard internal type theory of a presheaf category, $\mathcal{T}_{\widehat{\mathbb{C}}}$ from [AGH24, Section 3]. This follows from the fact that $\widehat{\mathbb{C}}$ is a reflective full subcategory of $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ (Proposition 4.2.4). These internal type theories are mediated by the functors in this adjunction.

4.4.20. *Remark.* When the regular part of the context is empty, as in judgements with the context

$$\Delta \vdash_\Delta$$

corresponding to an object

$$\begin{array}{c}\Delta\\ \big\|\\ \Delta\end{array}$$

in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$, the type theory $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ simplifies to the internal type theory of $\widehat{\mathbb{C}}_\flat$. The internal language $\mathcal{T}_{\widehat{\mathbb{C}}_\flat}$ is just the standard internal type theory of a presheaf category as detailed in [AGH24, Section 3]. This follows from the fact that $\widehat{\mathbb{C}}_\flat$ is a coreflective full subcategory of $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$, via the adjunction $\mathrm{cod} \dashv T$ from the fibred natural model in Theorem 4.1.3. These internal type theories are mediated by the functors in this adjunction.

5 Kripke-Joyal semantics for crisp type theory

Introduction

In Chapter 4, we developed a version of crisp type theory internal to the comma category $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ from the fibred model of dual-context type theory admitted by a presheaf category (Theorem 4.1.3). We now return to our motivating problem of relating diagrammatic and internal constructions of the universal uniform fibration by developing a tool for this task.

To precisely relate the category-theoretic and type-theoretic ways of reasoning in presheaf models of homotopy type theory, Awodey, Gambino, and Hazratpour [AGH24] generalise the Kripke-Joyal forcing semantics of the higher-order logic of a presheaf category $\widehat{\mathbb{C}}$ to its internal type theory $\mathcal{T}_{\widehat{\mathbb{C}}}$. Forcing for the internal type theory allows one to test the validity of type-theoretic judgements in the category. Awodey et al. show it to be a very convenient alternative to working directly with the interpretation of the internal language, and use it to study algebraic weak factorisation systems in models of homotopy type theory.

In this chapter, we follow [AGH24] to set-up the technique of Kripke-Joyal forcing for our internal crisp type theory $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ from Section 4.4. Firstly, we prove that there are *canonical generators* for the objects of the total category $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ of the fibred model from Chapter 4. That is, we give a result analogous to that for a presheaf category which says that every object is isomorphic to a colimit of representables. We then define the forcing condition and prove its basic properties.

5.1 CANONICAL GENERATORS

The forcing semantics in [AGH24] for the internal type theory of a presheaf category $\widehat{\mathbb{C}}$ relies on the fact that every presheaf is a colimit of representables. To define forcing semantics for the internal type theory of the comma category $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, we require an analogous result giving canonical generators for objects $\chi : X \to M$. To this end, we begin by recalling the precise result for presheaf categories.

5.1.1. PROPOSITION. *Let $\mathbb{C}$ be a small category, X be a presheaf over $\mathbb{C}$, and $\mathsf{el}(X)$ be the category of elements of X. Every X is naturally isomorphic to the colimit under the diagram*

$$\begin{aligned} \mathsf{el}(X) &\xrightarrow{\text{よ}\circ\pi} \widehat{\mathbb{C}} \\ (c,x) &\longmapsto \text{よ}c. \end{aligned}$$

That is,

$$X \cong \varinjlim_{(c,x)} \text{よ}c.$$

To adapt this to the case of the comma category $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ requires an analogous notion of *category of elements*, which we now define.

5.1.2. *Notation.* Recalling the fact that $\widehat{\mathbb{C}}_\flat$ is equivalent to Set (Proposition 4.1.4), for the remainder of this section we use the same notation for a set M and the constant presheaf on this set.

5.1.3. DEFINITION ("category of elements" for $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$). Let $\chi : X \to M$ be an object in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. The *category of elements* of χ, written $\mathsf{el}(\chi)$ is given as follows:

- an object is a triple
$$(c \in \mathsf{obj}(\mathbb{C}), x \in X(c), m \in M)$$
with $\chi_c(x) = m$
- an arrow is a triple of the form[1]
$$(f, x, m) : (d, y, n) \to (c, x, m)$$
where $f : d \to c$ is an arrow in $\mathbb{C}$ satisfying $X(f)(x) = y$, with the naturality of χ then implying $m = n$
- the identity arrow for an object (c, x, m) is (id_c, x, m)
- if $(f, x, m) : (d, y, n) \to (c, x, m)$ and $(g, y, n) : (d', y', n') \to (d, y, n)$, then the composite arrow
$$(f, x, m)\circ(g, y, n) : (d', y', n') \to (c, x, m)$$
is defined by
$$(f, x, m)\circ(g, y, n) := (f\circ g, x, m),$$
which satisfies the necessary identity $X(f\circ g)(x) = y'$.

[1]The notation (f, x, m) follows the requirement that an arrow determines its source and target – its source being $(\mathsf{dom}(f), X(f)(x), m)$ and its target being $(\mathsf{cod}(f), x, m)$.

It is straightforward to see that composition is associative and unital, and that $\mathsf{el}(\chi)$ is a small category for $\mathbb{C}$ small.

5.1.4. *Remark.* Using the Yoneda lemma, an object $(c, x, \chi_c(x))$ in $\mathsf{el}(\chi)$ corresponds to a morphism $(x, \chi_c(x))$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ given by the square

$$\begin{tikzcd}
よc \arrow[r, "x"] \arrow[d, "!"'] & X \arrow[d, "\chi"] \\
1 \arrow[r, "m"'] & M,
\end{tikzcd}$$

and an arrow $(f, x, m) : (d, y, n) \to (c, x, m)$ in $\mathsf{el}(\chi)$ corresponds to a morphism $(よf, \mathsf{id}_1)$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, as in

$$\begin{tikzcd}
よd \arrow[r, "よf"] \arrow[d] & よc \arrow[d] \\
1 \arrow[r, equals] & 1,
\end{tikzcd}$$

such that the following diagram commutes:

$$\begin{tikzcd}
よd \arrow[rr, "y"] \arrow[dr] \arrow[dd] & & X \arrow[dd, "\chi"] \\
& よc \arrow[ur, "x"] \arrow[dd] & \\
1 \arrow[rr, crossing over] \arrow[dr, equals] & & M \\
& 1. \arrow[ur] &
\end{tikzcd}$$

5.1.5. *Remark.* We want to consider the colimit of a diagram in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ indexed by $\mathsf{el}(\chi)$. To this end, define $P : \mathsf{el}(\chi) \to \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ on objects by

$$P(c, x, m) := \begin{array}{c} よc \\ \downarrow \\ 1 \end{array}$$

and on arrows by

$$P(f, x, m) := (よf, \mathsf{id}_1) : (よd, 1, !_{よd} : よd \to 1) \to (よc, 1, !_{よc} : よc \to 1),$$

given by the diagram

$$\begin{tikzcd}
よd \arrow[r, "よf"] \arrow[d] & よc \arrow[d] \\
1 \arrow[r, equals] & 1.
\end{tikzcd}$$

This is clearly functorial.

We can now state and prove our canonical generators result for the comma category $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$.

5.1.6. PROPOSITION. *For every object $\chi : X \to M$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, there is a natural isomorphism*

$$\begin{pmatrix} X \\ {\scriptstyle\chi}\big\downarrow \\ M \end{pmatrix} \simeq \varinjlim_{(c,x,m)} \begin{pmatrix} \text{よ}c \\ \big\downarrow \\ 1 \end{pmatrix}$$

for $(c, x, m) \in \mathsf{el}(\chi)$.

Proof. The $\mathsf{el}(\chi)$-indexed family of maps

$$\left(\begin{tikzcd} \text{よ}c \arrow[r, "x"] \arrow[d] & X \arrow[d, "\chi"] \\ 1 \arrow[r, "m"'] & M \end{tikzcd}\right)_{(C,x,m)\in\mathsf{el}(\chi)}$$

in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ gives a cocone under the diagram $P : \mathsf{el}(\chi) \to \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ from (Remark 5.1.5) with vertex $\chi : X \to M$, since for any arrow $(f, x, m) : (c', x', m) \to (c, x, m)$ in the index category, there is a commutative diagram

$$\begin{tikzcd} \text{よ}c' \arrow[rr, "x'"] \arrow[dd] \arrow[dr] & & X \arrow[dd, "\chi"] \\ & \text{よ}c \arrow[ur, "x"'] \arrow[dd] & \\ 1 \arrow[rr] \arrow[dr, equal] & & M \\ & 1 \arrow[ur] & \end{tikzcd}$$

by Remark 5.1.4. To show that this is a colimiting cocone, for any other cocone given by a vertex $\xi : Y \to N$ and an $\mathsf{el}(\chi)$-indexed family of maps

$$\left(\begin{tikzcd} \text{よ}c \arrow[r, "y_{(c,x,m)}"] \arrow[d] & Y \arrow[d, "\xi"] \\ 1 \arrow[r, "n_{(c,x,m)}"'] & N \end{tikzcd}\right)_{(c,x,m)\in\mathsf{el}(\chi)}$$

we must define a unique dashed morphism in the below diagram:

$$\begin{tikzcd} & \text{よ}c' \arrow[ddl] \arrow[ddd] \arrow[dr, "x'"] \arrow[drr, bend left, "y_{(c',x',m)}"] & & \\ & & X \arrow[r, dashed] \arrow[ddd] & Y \arrow[ddd] \\ \text{よ}c \arrow[urr, "x"] \arrow[urrr, bend right, "y_{(c,x,m)}"'] \arrow[ddd] & & & \\ & 1 \arrow[dr, "m"] \arrow[drr, bend left, "n_{(c',x',m)}"] \arrow[ddl, equal] & & \\ & & M \arrow[r, dashed] & N. \\ 1 \arrow[urr, "m"] \arrow[urrr, bend right, "n_{(c,x,m)}"'] & & & \end{tikzcd}$$

This $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$-morphism is given by a pair of natural transformations $(\alpha : X \to Y, \beta : M \to N)$ in $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{C}}_\flat$ respectively such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\chi"'] & Y \arrow[d, "\xi"] \\
M \arrow[r, "\beta"'] & N.
\end{tikzcd} \tag{5.1}$$

To define the component of α at an object c in $\mathbb{C}$, we use the Yoneda lemma to identify $x : \text{よ}c \to X$ with $x \in X(c)$ and $y_{(c,x,m)} : \text{よ}c \to Y$ with $y_{(c,x,m)} \in Y(c)$, and set

$$\alpha_c(x) := y_{(c,x,m)}.$$

Similarly, we define components of β by

$$\beta_c(m) := n_{(c,x,m)}.$$

It follows from the natural isomorphism of the Yoneda lemma that these are both natural in c. To see that the diagram in (5.1) commutes, we have the following evaluation at an object c in $\mathbb{C}$:

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & y_{(c,x,m)} \arrow[ddd, mapsto] \\
& X(c) \arrow[r, "\alpha_c"] \arrow[d, "\chi_c"'] & Y(c) \arrow[d, "\xi_c"] & \\
& M \arrow[r, "\beta_c"'] & N & \\
m \arrow[rrr, mapsto] & & & \beta_c(m)= n_{(c,x,m)}
\end{tikzcd}$$

□

5.2 The forcing condition and its basic properties

With this preliminary result in place, we proceed to the general definition and basic properties of forcing for the type theory $\mathcal{T}_{\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat}$. First, we introduce some useful notation.

5.2.1. *Notation.* To assist with working in the comma category in the proof of Proposition 5.2.3, and to make transparent the comparison with the results in [AGH24], we use "abbreviated" versions of diagrams in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. Noting that the objects in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ are arrows in $\widehat{\mathbb{C}}$, and that morphisms in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ are pairs of $\widehat{\mathbb{C}}$ arrows, we standardly use diagrams in $\widehat{\mathbb{C}}$ to reason in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. Instead, we will use diagams with nodes corresponding to obejcts in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ and arrows corresponding to morphisms in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. For example, the following diagram in $\widehat{\mathbb{C}}$

$$\begin{tikzcd}
& & E \arrow[dl, "\pi"'] \arrow[dd, "\varepsilon"] \\
X \arrow[urr, bend left, "a"] \arrow[r, "\alpha"'] \arrow[dd, "\chi"'] & U \arrow[dd] & \\
& & 1 \\
M \arrow[r] \arrow[urr, bend left] & 1 \arrow[ur, equal] &
\end{tikzcd}$$

which is used to represent the following equation between arrows in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$,

$$(\pi, \mathsf{id}_1)\circ(a, !_M) = (\alpha, !_M),$$

becomes

$$\begin{tikzcd}
 & \mathcal{E} \arrow[d, "\pi"] \\
\chi \arrow[ur, "a"] \arrow[r, "\alpha"'] & \mathcal{U}
\end{tikzcd}$$

where we also use Notation 4.4.2.

5.2.2. DEFINITION. Let $\chi : X \to M$ be an object in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. Let $\alpha : \chi \to \mathcal{U}$ and $(x, m) : !_{\text{よ}c} \to \chi$ be arrows in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, as in the diagram

$$\begin{tikzcd}
\text{よ}c \arrow[r, "x"] \arrow[d] & X \arrow[r, "\alpha"] \arrow[d, "\chi"'] & U \arrow[d] \\
1 \arrow[r, "m"'] & M \arrow[r] & 1.
\end{tikzcd}$$

(i) For $a : !_{\text{よ}c} \to \mathcal{E}$, we say that c *forces* $a : \alpha(x, m)$, written

$$c \Vdash a : \alpha(x, m),$$

if the following diagram commutes:

$$\begin{tikzcd}
 & \text{よ}c \arrow[rr, "a"] \arrow[dl, "x"'] \arrow[dd] & & E \arrow[dl, "\pi"] \arrow[dd] \\
X \arrow[rr, "\alpha"] \arrow[dd, "\chi"'] & & U \arrow[dd] & \\
 & 1 \arrow[rr] \arrow[dl, "m"] & & 1 \arrow[dl] \\
M \arrow[rr] & & 1. &
\end{tikzcd}$$

According to Notation 5.2.1, this abbreviates to the following diagram in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$:

$$\begin{tikzcd}
!_{\text{よ}c} \arrow[r, "a"] \arrow[d, "{(x,m)}"'] & \mathcal{E} \arrow[d, "\pi"] \\
\chi \arrow[r, "\alpha"'] & \mathcal{U}.
\end{tikzcd}$$

(ii) For $a, b : !_{\text{よ}c} \to \mathcal{E}$ such that $c \Vdash a : \alpha(x, m)$ and $c \Vdash b : \alpha(x, m)$, we say that c *forces* $a = b : \alpha(x, m)$, written

$$c \Vdash a = b : \alpha(x, m),$$

if a and b are equal maps in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$.

We now relate the validity of a judgement $X \vdash_M a : \alpha$ in the internal type theory $\mathcal{T}_{\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat}$ to the forcing semantics. This is an analogue of [AGH24, Proposition 4.5].

5.2.3. PROPOSITION. *Let $\chi : X \to M$ be an object in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ and let $X \vdash_M \alpha : U$ be a judgement in $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$. Suppose that for each $(x, m) : !_{\text{よ}c} \to \chi$ there is a given element $a_{(x,m)} : !_{\text{よ}c} \to \mathcal{E}$ such that*

$$c \Vdash a_{(x,m)} : \alpha(x, m), \tag{5.2}$$

and moreover, the following uniformity condition holds for all $f : !_{\text{よ}c} \to !_{\text{よ}d}$,

$$d \Vdash a_{(x,m)}(f) = a_{(xf,m)} : \alpha(xf, m). \tag{5.3}$$

Then there is a unique element $a : \chi \to \mathcal{E}$ such that

$$X \vdash_M a : \alpha \tag{5.4}$$

and, for all $(x, m) : !_{\text{よ}c} \to \chi$,

$$c \Vdash a(x, m) = a_{(x,m)} : \alpha(x, m). \tag{5.5}$$

Proof. Suppose that the judgement $X \vdash_M \alpha : \mathcal{U}$ satisfies the premise of the proposition. Translating typing judgement (5.4) and forcing condition (5.5) into diagrams, we want to construct a unique arrow a in the diagram

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "\pi"] \\ \chi \arrow[r, "\alpha"'] \arrow[ur, "a"] & \mathcal{U} \end{tikzcd}$$

such that the following commutes

$$\begin{tikzcd} & & \mathcal{E} \arrow[d, "\pi"] \\ !_{\text{よ}c} \arrow[r, "{(x,m)}"'] \arrow[urr, bend left, "a_{(x,m)}"] & \chi \arrow[r, "\alpha"'] \arrow[ur, "a"] & \mathcal{U} \end{tikzcd}$$

for every (x, m) and its associated element $a_{(x,m)}$. To construct such an a, observe that the uniformity condition (5.3) translates to the diagram

$$\begin{tikzcd} & & & \mathcal{E} \arrow[d, "\pi"] \\ !_{\text{よ}d} \arrow[r, "f"'] \arrow[urrr, bend left, "a_{(xf,m)}"] & !_{\text{よ}c} \arrow[r, "{(x,m)}"'] \arrow[urr, bend left, "a_{(x,m)}"] & \chi \arrow[r, "\alpha"'] & \mathcal{U}, \end{tikzcd}$$

from which we know that the following diagram commutes

$$\begin{tikzcd} !_{\text{よ}d} \arrow[dd, "f"'] \arrow[dr, "{(xf,m)}"] \arrow[drr, bend left, "a_{(xf,m)}"] & & \\ & \chi & \mathcal{E} \\ !_{\text{よ}c} \arrow[ur, "{(x,m)}"'] \arrow[urr, bend right, "a_{(x,m)}"'] & & \end{tikzcd}$$

for any f. The uniform family

$$(a_{(x,m)} : !_{\text{よ}c} \to \mathcal{E})$$

indexed by the full subcategory $(!_{ⅎ})/\chi \hookrightarrow \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat/\chi$ can be indexed by the equivalent category $\mathsf{el}(\chi)$, and so gives a cocone with vertex $\mathcal{E}$ under the diagram $P : \mathsf{el}(\chi) \to \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$. By Proposition 5.1.6, there is a unique map $\chi \to \mathcal{E}$ since χ is the colimiting cocone. □

The following corollary summarises Proposition 5.2.3.

5.2.4. COROLLARY. *Let $\alpha : \chi \to \mathcal{U}$. Then the following data are in bijective correspondence:*

(i) elements $a : \chi \to \mathcal{E}$ such that $X \vdash_M a : \alpha$,

(ii) uniform families of elements $a_{(x,m)} : !_{ⅎ c} \to \mathcal{E}$ such that $c \Vdash a_{(x,m)} : \alpha(x,m)$.

Specifically, the bijection is defined by letting $a_{(x,m)} := a(x,m) : \alpha(x,m)$.

The results in this chapter are the foundation for forcing semantics for the internal type theory $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ from Chapter 4. The next step in developing these semantics is to unfold the definition of forcing with respect to each of the type-forming operations of $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$. We postpone this to future work, for the following reason.

5.2.5. *Remark.* In the next chapter (Chapter 6), we address our initial goal of precisely relating the diagrammatic and type-theoretic constructions of the universal uniform fibration. We discover that working directly with the interpretation of the internal language in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ is sufficient for the portion of the construction that we consider in Theorem 6.3.8. We anticipate, however, that we will find uses for the forcing semantics for $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ if we consider other aspects of the construction in future work.

6 Universal uniform fibration in presheaf-based models

Introduction

The structure of a *uniform fibration* on a map in a presheaf category models the higher structure of a type in homotopy type theory. Uniform fibrations were first defined in the setting of categories of *cubical sets* [BCH14; CCHM18] before being analysed as a part of *algebraic weak factorisation systems* (awfs's for short) in [Swa16] and then generalised to the setting of a presheaf category equipped with a class of cofibrations and an interval object [GS17; Awo23]. Developing the theory of awfs's using the internal type theory of a presheaf category was suggested in [Coq15] as a convenient way of handling complex categorical stucture and enabling formalisations in interactive theorem provers, and was realised in [OP18]. These category-theoretic and type-theoretic descriptions have since been reconciled in [AGH24] using a version of Kripke-Joyal forcing for the internal type theory of a presheaf category.

Our focus in this chapter is the notion of a *universal* or *classifying* uniform fibration, analogous to the small map classifier in a presheaf category given by the Hofmann-Streicher universe (Example 2.1.11). Such a map is constructed in [CCHM18], specifically a version of the Hofmann-Streicher universe equipped with a fibration structure and from which every fibration can be obtained by pullback. It is not included, however, in related work in an internal language [OP16; BBC+19]; in fact, it is shown in [OP18, Remark 7.5] that there can be no version of the universal uniform fibration in the standard internal type theory of a presheaf category.

The problem of constructing an internal version of the universal uniform fibration is addressed in [LOPS18] by working in a different internal language. *Crisp type theory* refers to the fragment of Shulman's spatial type theory [Shu18] singled-out for this purpose. It remains, however, to precisely relate the category-theoretic and type-theoretic constructions of the universal uniform fibration. Although [AGH24] reconcile these two styles of descriptions for uniform fibrations, they note that their construction of the universal uniform fibration (following [Coq17; LOPS18; Awo23]) involves a mix of category theory and internal language, but crucially cannot be done purely in the internal language. The limitation is that their Kripke-Joyal forcing is developed for the standard extensional type theory internal to a presheaf category and so does not immediately apply to the type theory used in [LOPS18]. Furthermore, there is not yet a counterpart of the standard semantics of extensional type theory [Hof97] for the type theory of [LOPS18] that allows one to unfold the interpretation of the internal type theory into the presheaf category.

The goal of this chapter is to go some way towards precisely relating the category-theoretic and type-theoretic constructions of the universal uniform fibration—the former in [AGH24] and the latter in [LOPS18]—using the work of Chapter 4, specifically the theory of the comma category $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ developed in Section 4.2 and the crisp internal language extracted in Section 4.4. We begin by recalling why a universal uniform fibration cannot be constructed in the standard internal type theory of a presheaf category (Section 6.1). Next, we give a diagrammatic construction of the universal uniform fibration (Section 6.2) following the type-theoretic construction in [LOPS18], with comparison to the proof in [AGH24]. In Section 6.3, we precisely relate parts of the category-theoretic proof in Section 6.2 to the type-theoretic proof in [LOPS18]. In this way, we improve or expand on the two existing proofs: compared to [LOPS18], we have a notion of model, and compared to [AGH24], we are not limited by working in the ordinary extensional type theory of a presheaf category.

6.1 Preliminaries: internal adjunctions, tiny interval in a presheaf category

A universal uniform fibration cannot be constructed in the internal type theory of a presheaf category $\widehat{\mathbb{C}}$ because a certain adjunction involving the interval object in $\widehat{\mathbb{C}}$ does not internalise. To explain this problem, we begin by presenting the notion of *internal adjunction* as described in [LOPS18]. We make explicit its relationship to standard adjunctions in Proposition 6.1.3 and, in the setting of presheaves, we give a characterisation with respect to generalised elements in Lemma 6.1.6. We then specialise to the general setting in which uniform fibrations can be defined: a presheaf category with a *class of cofibrations* and an *interval object*. Next we state the property of *tinyness* of the interval, which is used in [LOPS18] to construct the universal uniform fibration and says that a particular adjunction exists. Finally, we will see that this adjunction does internalise.

Internal adjunctions

Let $\mathcal{C}$ be a cartesian closed category, i.e. a category with finite products (including a terminal object) and exponentials. Consider a pair of endofunctors $F, G : \mathcal{C} \to \mathcal{C}$. We say that F is left adjoint to G when there are isomorphisms

$$\mathcal{C}(Fc, d) \cong \mathcal{C}(c, Gd) \tag{6.1}$$

for each $c, d \in \mathcal{C}$ that are natural in both variables.

We wish to consider an internal notion of adjunction between endofuntors. Given F, G, and an object $d \in \mathcal{C}$, we can define endofunctors

$$d^{F(-)} : \mathcal{C} \to \mathcal{C} \text{ and } (Gd)^{(-)} : \mathcal{C} \to \mathcal{C}$$

that take an object c in $\mathcal{C}$ to the exponential objects d^{Fc} and $(Gd)^c$ respectively. Then we can make the following definition.

6.1.1. DEFINITION (internal adjunction). An *internal adjunction* in $\mathcal{C}$ consists of a pair of endofunctors $F, G : \mathcal{C} \to \mathcal{C}$ together with a family of natural isomorphisms

$$\alpha_d : d^{F(-)} \xrightarrow{\cong} (Gd)^{(-)} \tag{6.2}$$

for $d \in \mathcal{C}$.

6.1.2. *Remark.* This is an ad hoc notion merely intended to subsume the phenomenon described in [LOPS18]. The precise connection to enriched notions of adjunction, for example, is not pursued here.

The adjunction in Definition 6.1.1 is "internal" in the sense that it replaces maps between objects with exponential objects. Internal adjunctions give rise to standard adjunctions in the following way.

6.1.3. PROPOSITION. *Suppose there is an internal adjunction given by functors $F, G : \mathcal{C} \to \mathcal{C}$ and a family of natural isomorphisms $\alpha_d : d^{F(-)} \to (Gd)^{(-)}$ for $d \in \mathcal{C}$. Then F is left adjoint to G.*

Proof. A natural isomorphism α_d gives rise to isomorphisms on global elements

$$\mathcal{C}(1, d^{Fc}) \cong \mathcal{C}(1, (Gd)^c),$$

for each $c \in \mathcal{C}$, by postcomposition with α_c. These isomorphisms are natural in both variables. Transposing both sides over the product-exponent adjunction $(-) \times x \dashv (-)^x$ gives the natural isomorphism

$$\mathcal{C}(1 \times Fc, d) \cong \mathcal{C}(1 \times c, Gd)$$

which is equivalent to the required isomorphism in (6.1). □

6.1.4. *Observation.* The converse to Proposition 6.1.3 is not necessarily true because the first step of the proof is not reversible: isomorphisms on global elements do not give rise to the required family of natural isomorphisms. An example of this is given in Observation 6.1.15.

Internal adjunctions in a presheaf category

We now specialise to internal adjunctions in a presheaf category $\widehat{\mathbb{C}}$, for some small category $\mathbb{C}$ with a terminal object.

6.1.5. *Notation.* We adopt the notation $X \Rightarrow Y$ for the exponential object Y^X, as this will be more suitable for working with iterated exponents.

We have the following characterisation of internal adjunctions in $\widehat{\mathbb{C}}$.

6.1.6. LEMMA. *Let F and G be endofunctors on $\widehat{\mathbb{C}}$. There is an internal adjunction between F and G if and only if there is an isomorphism*

$$\widehat{\mathbb{C}}(\text{よ}_c \times FX, Y) \cong \widehat{\mathbb{C}}(\text{よ}_c \times X, GY) \tag{6.3}$$

for each $X, Y \in \widehat{\mathbb{C}}$ that is natural in both variables.

Proof. An internal adjunction is given by a family of natural isomorphisms $F(-) \Rightarrow Y \cong (-) \Rightarrow GY$ for $Y \in \widehat{\mathbb{C}}$. In a presheaf category, this is equivalent to having isomorphisms

$$(FX \Rightarrow Y)(c) \cong (X \Rightarrow GY)(c) \tag{6.4}$$

in Set for every $X \in \widehat{\mathbb{C}}$, natural in X, and for every $c \in \mathbb{C}$. By the Yoneda lemma, such an isomorphism is equivalent to one of the form

$$\widehat{\mathbb{C}}(\text{よ}_c, FX \Rightarrow Y) \cong \widehat{\mathbb{C}}(\text{よ}_c, X \Rightarrow GY). \tag{6.5}$$

Transposing both sides over product-exponent adjunctions gives the required isomorphism. □

6.1.7. *Remark.* Informally, one could say that the exponents $FX \Rightarrow Y$ and $X \Rightarrow GY$ agree on generalised elements in an internalisable adjunction $F \dashv G$, while they agree only on global elements in a general (not necessarily internalisable) adjunction.

6.1.8. EXAMPLE. In a presheaf category $\widehat{\mathbb{C}}$, the product-exponent adjunction $(-) \times Z \dashv Z \Rightarrow (-)$ can be internalised. We have the isomorphism

$$\widehat{\mathbb{C}}(\text{よ}_c \times (X \times Z), Y) \cong \widehat{\mathbb{C}}(\text{よ}_c \times X, Z \Rightarrow Y), \tag{6.6}$$

since maps on the right-hand side are exponential transposes of maps on the left-hand side, so the adjunction is internalisable by Lemma 6.1.6.

Tiny interval in a presheaf category

We continue in the setting of a presheaf category $\widehat{\mathbb{C}}$ and introduce the conditions necessary for defining uniform fibrations and thus a model of homotopy type theory in cubical sets, as in [BCH14; CCHM18]. The first is that $\widehat{\mathbb{C}}$ has a *class of cofibrations*, which we define following [AGH24, Section 5], writing $\widehat{\mathbb{C}}^{\rightarrow}_{\text{cart}}$ for the category of arrows and pullback squares in $\widehat{\mathbb{C}}$.

6.1.9. DEFINITION. A *class of cofibrations* in $\widehat{\mathbb{C}}$ is a class of maps Cof satisfying the following conditions:

(i) the elements of Cof are monomorphisms
(ii) the unique map $0 \to 1$ from the initial object to the terminal object is in Cof
(iii) the identity map $\mathsf{id}_1 : 1 \to 1$ is in Cof
(iv) Cof is stable under pullback along all maps
(v) Cof is closed under composition
(vi) the full subcategory of $\widehat{\mathbb{C}}^{\rightarrow}_{\text{cart}}$ spanned by Cof has a terminal object.

6.1.10. *Terminology.* An object X in $\widehat{\mathbb{C}}$ is said to be *cofibrant* if the unique map $!_X : X \to 1$ is a cofibration.

The second requirement is an *interval object*, used for modelling path types. An interval is an object $\mathbb{I}$ in $\widehat{\mathbb{C}}$ with distinct endpoints, that is, maps $\delta^k : 1 \to \mathbb{I}$ for $k = 0, 1$ with $\delta^0 \neq \delta^1$. For any object X in $\widehat{\mathbb{C}}$, a map $x : \mathbb{I} \to X$ is thought of as a path in X. Then for any $\alpha : X \to U$, maps $\alpha \circ x : \mathbb{I} \to U$ are thought of as restrictions of the type α along paths in X.

6.1.11. *Notation.* Given the intuition that maps $\alpha \circ x : \mathbb{I} \to U$ for $x : \mathbb{I} \to X$ and $\alpha : X \to U$ are restrictions of α along paths in X, we view $\alpha \circ x$ as a map from the exponent $X^{\mathbb{I}}$ and write

$$\alpha \circ x : X^{\mathbb{I}} \to U.$$

This understanding of the interval is sufficient for our purposes, but for completeness, we include the definition of interval with connections from [AGH24].

6.1.12. DEFINITION (interval with connections [AGH24, Definition 8.1]). An *interval with connections* is an object $\mathbb{I}$ in $\widehat{\mathbb{C}}$ equipped with endpoints $\delta^k : 1 \to \mathbb{I}$ and connections $c_k : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ for $k = 0, 1$, satisfying the following axioms:

(i) the maps $\delta^k : 1 \to \mathbb{I}$ are cofibrations, for $k = 0, 1$

(ii) the pullback of δ^0 and δ^1 is the initial object 0,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\delta^1"] \\ 1 \arrow[r, "\delta^0"'] & \mathbb{I} \end{tikzcd}$$

(iii) the diagrams

$$\begin{tikzcd} \mathbb{I} \arrow[r, "{(\delta^k,1)}"] \arrow[d] & \mathbb{I} \times \mathbb{I} \arrow[d, "c_k"] & \mathbb{I} \arrow[l, "{(1,\delta^k)}"'] \arrow[d] \\ 1 \arrow[r, "\delta^k"'] & \mathbb{I} & 1 \arrow[l, "\delta^k"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{I} \arrow[r, "{(\delta^{1-k},1)}"] \arrow[dr, equal] & \mathbb{I} \times \mathbb{I} \arrow[d, "c_k"] & \mathbb{I} \arrow[l, "{(1,\delta^{1-k})}"'] \arrow[dl, equal] \\ & \mathbb{I} & \end{tikzcd}$$

commute, for $k = 0, 1$.

For the construction of the universal uniform fibration, we require that the interval is *tiny*.

6.1.13. DEFINITION (tiny interval). An interval object $\mathbb{I}$ is said to be *tiny* if the exponentiation functor $(-)^{\mathbb{I}}$ has a right adjoint. We denote this right adjoint by $\sqrt{(-)}$ and call it the *root functor*.

6.1.14. EXAMPLE (presheaf category on cubical sets). When $\mathbb{C}$ is one of various *cube categories* used for cubical models of type theory [BCH14; ABC+19; CCHM18], denoted $\square$, the presheaf category $\widehat{\square}$ has a tiny interval. Following [LOPS18, Definition 2.1], let $\square$ denote the small category with finite products which is the Lawvere theory of de Morgan algebra. The *generic de Morgan algebra* is an object I in $\square$ with two distinct global elements, sent by the Yoneda embedding $\text{よ} : \square \hookrightarrow \widehat{\square}$ to a representable presheaf $\mathbb{I} := \text{よ}I$ satisfying the conditions of Definition 6.1.12. This interval object is also tiny (see, for example, [LOPS18, Section 5]). Briefly, the exponentiation functor $(\_)^{\mathbb{I}} : \widehat{\square} \to \widehat{\square}$ is naturally isomorphic to the endofunctor

$$(\_ \times I)^* : \widehat{\square} \to \widehat{\square}$$

induced by precomposition with $\_ \times I : \square \to \square$. But $(\_ \times I)^*$ has left and right adjoints given by Kan extension, so the interval is tiny.

6.1.15. *Observation.* In a category of cubical sets $\widehat{\square}$, as in Example 6.1.14, the adjunction $(-)^{\mathbb{I}} \dashv \sqrt{(-)}$ is not internalisable (see [LOPS18, Theorem 5.1]). This would require isomorphisms of the form

$$\widehat{\square}(\text{よ}_c \times X^{\mathbb{I}}, Y) \cong \widehat{\square}(\text{よ}_c \times X, \sqrt{Y}),$$

but we only have
$$\widehat{\square}(1, A^{\mathbb{I}} \Rightarrow B) \cong \widehat{\square}(1, A \Rightarrow \sqrt{B}).$$

6.2 CONSTRUCTION OF THE UNIVERSAL UNIFORM FIBRATION

Let $\widehat{\mathbb{C}}$ be a presheaf category for some small category $\mathbb{C}$ and suppose that $\widehat{\mathbb{C}}$ has a class of cofibrations Cof and a tiny interval $\mathbb{I}$. In this setting, the types of homotopy type theory are modelled by certain kinds of maps called *uniform fibrations*. A uniform fibration structure on a small map $p : A \to X$ in $\widehat{\mathbb{C}}$ is given by a function j assigning a *filler* to certain diagrams, plus a *uniformity condition*. Looking at the definition of a uniform fibration structure [AGH24, Definition 8.2] one sees how it could be more convenient to package the data as a type, rather than carry around the function j and the conditions on fillers.

Just as it is convenient to work with a classifier of small maps $\mathsf{ty} : E \to U$ (Example 2.1.11), we wish to have a classifier of (small) uniform fibrations. Such a classifier cannot be constructed in the internal type theory of a presheaf category because it requires the adjunction $(-)^{\mathbb{I}} \dashv \sqrt{(-)}$, which does not internalise. Licata et al. [LOPS18] overcome this problem by using crisp type theory as the internal language, a dual-context type theory in which the first context zone comprises global elements and the second the usual local elements. Local elements may depend on global elements but not vice versa. Restricting to global elements, it is possible to have an internal version of the adjunction in question, specified by axioms we consider in Section 6.3.

In this section, we provide a synthesis of the proofs in [LOPS18] and [AGH24], making explicit how these two kinds of reasoning are combined. We observe that while one proof is done internally and the other externally, they both use a mixture of diagrammatic and type-theoretic reasoning. We begin with the preliminary definitions, then construct a candidate universal uniform fibration and prove that it is indeed a fibration that classifies.

6.2.1. *Remark.* While we interchangeably say that a map "has a fibration structure" or "is a fibration", we note that we are referring to structure on the map, as opposed to a property of the map.

Uniform fibrations

The notion of uniform fibration introduced in [CCHM18] is generalised in [OP18] from a specific presheaf model to any topos with an interval object. These respectively category-theoretic and type-theoretic descriptions are related in [AGH24, Section 8], in which a uniform fibration is formulated in the internal type theory of $\widehat{\mathbb{C}}$ in two different ways: with respect to trivial fibrations [AGH24, Proposition 8.5] and with respect to *fillings* [AGH24, Corollary 8.7]. Following the second method, we begin by recalling several definitions from [AGH24, Section 8].

Firstly, a fibration structure is parametrised by a *filling structure* on the universe of small maps. In this way, the theory is developed independently of a specific notion of filling.

6.2.2. DEFINITION (filling structure). A *filling structure* is a map
$$\mathsf{Fill} : U^{\mathbb{I}} \to U.$$

Let X be an object and $\alpha : X \to U$ be a family of types. A *filling structure on* α, denoted $\mathsf{Fill}(\alpha\circ-)$, is the composition

$$X^{\mathbb{I}} \xrightarrow[\alpha\circ-]{} U^{\mathbb{I}} \xrightarrow[\mathsf{Fill}]{} U, \tag{6.7}$$

where $\alpha\circ- : X^{\mathbb{I}} \to U^{\mathbb{I}}$ is defined by restricting α along paths $x : \mathbb{I} \to X$ in X, as in

$$\mathbb{I} \xrightarrow[x]{} X \xrightarrow[\alpha]{} U.$$

6.2.3. *Notation.* We take liberty with the notation for function spaces, identifying an element of the function type $X^{\mathbb{I}}$ with a map $x : \mathbb{I} \to X$.

6.2.4. DEFINITION (type of fibration structures). Let $\alpha : X \to U$. The type of *fibration structures for* α is the object

$$\mathsf{isFib}(\alpha) := \Pi_{x:X^{\mathbb{I}}}\mathsf{Fill}(\alpha\circ x). \tag{6.8}$$

6.2.5. *Remark.* This is the same as Awodey et al.'s *Fib* [AGH24, Equation 8.9] and Licata et al.'s *isFib* [LOPS18, Definition 2.2]. Awodey et al. refer to this as the *classifier of uniform fibration structures on a type*. The map that classifies fibrations, in an analogous sense to which ty classifies small maps, is called the *universe of uniform fibrations*.

6.2.6. *Remark.* Recalling that there are three equivalent notions of a small map $p : A \to X$, its small map classifier $\alpha : X \to U$, and the display map $p_\alpha : X.\alpha \to X$ (Remark 2.1.12), we equivalently talk about a fibration structure on p, on α, or on p_α.

6.2.7. EXAMPLE (CCHM fibration). The Cohen-Coquand-Huber-Mörtberg (CCHM) notion of fibration from their cubical sets model [CCHM18] is expressed in the internal type theory of a presheaf category equipped with a class of cofibrations and a suitable interval object in [OP18]. We present the internal formulation from [OP18, Definition 5.6], in the notation of [AGH24, Equation 8.8]. Fillings are parametrised by interval endpoints, so for an $\mathbb{I}$-indexed family of types $\alpha : \mathbb{I} \to U$, we have the following definition of *0-directed filling structure*:

$$\mathsf{Fill}_0(\alpha) := (\Pi_{\varphi:\Phi}\Pi_{v:\{\varphi\}\to\Pi_{i:\mathbb{I}}\alpha_i}\Pi_{a:\alpha_0}(\tilde{v}_0 = \lambda a)) \to \Sigma_{s:\Pi_{i:\mathbb{I}}\alpha_i}(s_0 =_{\alpha_0} a) \times (v = \lambda s).$$

Unpacking this type, a filling operation is concerned with extending a morphism $p : A \to X$ along a cofibration $m : S \rightarrowtail X$. Internally, this corresponds to extending cofibrant partial elements to totally defined elements. The operation of filling from 0 thus takes any cofibrant partial path (ϕ, v), along with an element a extending the partial path's evaluation at 0, and extends it to a dependently typed path $s : \Pi_{i:\mathbb{I}}\alpha_i$ that agrees with a at 0. See [AGH24, Section 8] for a full explanation of this notation and [OP18, Section 5] for more detail.

We use Fill_0 and Fill_1—defined in the same way as Fill_0 but with every 0 replaced by a 1—to define a map $\mathsf{Fill} : U^{\mathbb{I}} \to U$ as follows. For any object X and family of types $\alpha : X \to U$, we obtain an $\mathbb{I}$-indexed family of types $x : X^{\mathbb{I}} \vdash \alpha\circ x : \mathbb{I} \to U$ by restricting α along paths in X. A filling structure comes from having both 0- and 1-directed filling structures:

$$\mathsf{Fill}(\alpha\circ-) := \mathsf{Fill}_0(\alpha\circ-) \times \mathsf{Fill}_1(\alpha\circ-).$$

6.2.8. *Remark.* In [LOPS18], the filling structure is replaced by the simpler notion of *composition structure*, which is also a map $U^{\mathbb{I}} \to U$ but one that produces an extension at one end of a cofibrant partial path from an extension at the other. For CCHM fibrations, every filling structure gives rise to a composition structure and vice versa. While this simplifies certain developments in [CCHM18]—and their internal counterparts in [OP18]—there is no benefit for the present work. Therefore, we parametrise the type of fibrations by the more general notion of filling structure, so as to encompass other notions of fibration where a filling structure cannot be replaced by a composition structure (such as [BCH14]).

We now recall the result of Awodey et al. [AGH24] establishing that the object $\mathsf{isFib}(\alpha)$ is equivalent to the standard, diagrammatic notion of fibration structure.

6.2.9. PROPOSITION ([AGH24, Propositions 8.5 and 8.6]). *Let $\alpha : X \to U$. Then the following conditions are equivalent:*

(i) The display map $p : X.\alpha \to X$ admits a uniform fibration structure.
(ii) There is a term of type $\mathsf{isFib}(\alpha)$.

The predicate of fibration structures on a type was defined using an expression in the internal type theory of $\widehat{\mathbb{C}}$. We unfold what this corresponds to diagrammatically in the following lemma.

6.2.10. LEMMA. *Let $p : A \to X$ be a small map classified by $\alpha : X \to U$. A uniform fibration structure on p (equivalently on α or on $p_\alpha : X.\alpha \to X$) is given by a section of $p_{\mathsf{Fill}(\alpha\circ -)} : X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ -) \to X^{\mathbb{I}}$,*

$$\begin{tikzcd} X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ -) \arrow[d] \\ X^{\mathbb{I}}, \arrow[u, bend left, "f"] \end{tikzcd} \tag{6.9}$$

which is determined by the diagram

$$\begin{tikzcd} X^{\mathbb{I}} \arrow[dr, dashed, "f=\langle \mathsf{id}, f'\rangle"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "\mathsf{id}"'] & & \\ & X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ -) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "ty"] \\ & X^{\mathbb{I}} \arrow[r, "\mathsf{Fill}(\alpha\circ -)"'] & U. \end{tikzcd} \tag{6.10}$$

Proof. By Proposition 6.2.9, a fibration structure is given by a term of the closed type $\mathsf{isFib}(\alpha)$, that is, a global section

$$\begin{tikzcd} \Pi_{x:X^{\mathbb{I}}}\mathsf{Fill}(\alpha\circ x) \arrow[d] \\ 1. \arrow[u, bend left, "g"] \end{tikzcd} \tag{6.11}$$

By the characterisation of sections of a display map of a dependent product in [AGH24, Proposition 1.5], this is in bijective correspondence with sections as in (6.9). The second part of the lemma is just the universal property of a pullback and will be useful for later proofs. □

Finally, we prove reindexing lemmas for filling structures and fibration structures, to be used in the next section.

6.2.11. LEMMA. *Let $\alpha : X \to U$ and $s : Y \to X$. A filling structure on α can be reindexed along s to give a filling structure on $\alpha[s] : Y \to U$, with*

$$\mathsf{Fill}(\alpha\circ-)\circ s^{\mathbb{I}} = \mathsf{Fill}(\alpha[s]\circ-). \tag{6.12}$$

Proof. The action of $(-)^{\mathbb{I}}$ on morphisms is by postcomposition, meaning for any path $y : \mathbb{I} \to Y$, $s^{\mathbb{I}}(y) = s\circ y$. The lemma then follows from the associativity of composition:

$$\begin{aligned}(\mathsf{Fill}(\alpha\circ-)\circ s^{\mathbb{I}})(y) &= \mathsf{Fill}(\alpha\circ-)(s\circ y)\\ &= \mathsf{Fill}(\alpha\circ(s\circ y))\\ &= \mathsf{Fill}((\alpha\circ s)\circ y)\\ &= \mathsf{Fill}(\alpha[s]\circ y).\end{aligned}$$

□

6.2.12. LEMMA. *Let $\alpha : X \to U$ and $s : Y \to X$. A fibration structre on α given by a section $f : X^{\mathbb{I}} \to X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ x)$ over $X^{\mathbb{I}}$ can be reindexed along s to get a fibration structure $f\circ s^{\mathbb{I}}$ on $\alpha[s] : Y^{\mathbb{I}} \to U$.*

Proof. A fibration structure on $\alpha[s]$ is given by a section $Y^{\mathbb{I}} \to Y^{\mathbb{I}}.\mathsf{Fill}(\alpha[s]\circ-)$ over $Y^{\mathbb{I}}$. By Lemma 6.2.11, this is the same as a section of $Y^{\mathbb{I}} \to Y^{\mathbb{I}}.(\mathsf{Fill}(\alpha\circ-)\circ s^{\mathbb{I}})$ over $Y^{\mathbb{I}}$. From the universal property of pullbacks in the below diagram, it follows that $f\circ s^{\mathbb{I}}$ is a section:

$$\begin{tikzcd} Y^{\mathbb{I}}.(\mathsf{Fill}(\alpha\circ-)\circ s^{\mathbb{I}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ Y^{\mathbb{I}} \arrow[r, "s^{\mathbb{I}}"'] \arrow[u, bend left, "f\circ s^{\mathbb{I}}"] & X^{\mathbb{I}} \arrow[r, "\mathsf{Fill}(\alpha\circ-)"'] \arrow[u, bend left, "f"] & U. \end{tikzcd}$$

□

Candidate for a universal uniform fibration

We now construct a candidate for a universal uniform fibration. Consider a generic type $x : U \vdash x : U$, interpreted by the identity map $\mathrm{id} : U \to U$. A filling structure (Definition 6.2.2) on a generic type is a map

$$U^{\mathbb{I}} \xrightarrow[\mathsf{Fill}(\mathrm{id}\circ-)]{} U,$$

which may be transposed across the adjunction $(-)^{\mathbb{I}} \dashv \sqrt{(-)}$ to give

$$U \xrightarrow[\widetilde{\mathsf{Fill}(\mathrm{id}\circ-)}]{} \sqrt{U}.$$

Our candidate universe of fibrant types, U_{Fib}, is the result of applying the root functor to the small map classifier ty and pulling back along $\widetilde{\mathsf{Fill}(\mathrm{id}\circ-)}$:

$$\begin{tikzcd} U_{\mathsf{Fib}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sqrt{E} \arrow[d, "\sqrt{\mathsf{ty}}"] \\ U \arrow[r, "\widetilde{\mathsf{Fill}(\mathrm{id}\circ-)}"'] & \sqrt{U}. \end{tikzcd} \tag{6.13}$$

The candidate classifying map, $\mathsf{ty}_{\mathsf{Fib}}$, is then obtained by pullback of the small map classifer along the "typing map" $\pi_1 : U_{\mathsf{Fib}} \to U$:

$$\begin{tikzcd} E_{\mathsf{Fib}} \arrow[r] \arrow[d, "\mathsf{ty}_{\mathsf{Fib}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\mathsf{ty}"] \\ U_{\mathsf{Fib}} \arrow[r, "\pi_1"'] & U. \end{tikzcd} \tag{6.14}$$

6.2.13. *Terminology.* We call U_{Fib} the *universe of (small) fibrant types*, analogously to U being the universe of small types; we call $\mathsf{ty}_{\mathsf{Fib}}$ the *universal uniform fibration* or *classifier of uniform fibrations*, analogously to ty being the classifier of small types.

6.2.14. *Observation.* The above construction of the universal uniform fibration follows Licata et al. [LOPS18, Theorem 5.2]. We can see that it is the same as the construction by Awodey et al. [AGH24] as follows. Awodey et al. construct a pullback stable family $\mathsf{Fib}^*(\alpha)$ over X by applying the root functor to $\mathsf{Fill}(\alpha\circ-)$, viewed as a display map over $X^{\mathbb{I}}$, along the unit of the adjunction ([AGH24, Diagram 8.14]):

$$\begin{tikzcd} \mathsf{Fib}^*(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sqrt{X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ-)} \arrow[d] \\ X \arrow[r, "\eta"'] & \sqrt{X^{\mathbb{I}}}. \end{tikzcd} \tag{6.15}$$

The universe of small fibrations is then defined in [AGH24, Theorem 8.9] to be $\mathsf{Fib}^*(\mathrm{id})$, as in the left-hand pullback in the following diagram:

$$\begin{tikzcd} U_{\mathsf{Fib}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sqrt{U^{\mathbb{I}}.\mathsf{Fill}(\mathrm{id}\circ-)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sqrt{E} \arrow[d] \\ U \arrow[r, "\eta"] \arrow[rr, bend right, "\widetilde{\mathsf{Fill}(\mathrm{id}\circ-)}"'] & \sqrt{U^{\mathbb{I}}} \arrow[r, "\sqrt{\mathsf{Fill}(\mathrm{id}\circ-)}"] & \sqrt{U}. \end{tikzcd}$$

Licata et al.'s construction is the outer diagram, noting that this is a pullback because the right-hand square is a pullback, since the root functor is a right adjoint and so preserves limits.

Uniform fibrations are classified by $\mathsf{ty}_{\mathsf{Fib}}$

We prove that the candidate universal uniform fibration is in fact a universal uniform fibration, first by showing that it is a uniform fibration and then by showing the sense in which it is universal.

6.2.15. PROPOSITION. *The map* $\mathsf{ty}_{\mathsf{Fib}} : E_{\mathsf{Fib}} \to U_{\mathsf{Fib}}$ *defined in (6.13) and (6.14) is a uniform fibration.*

Proof. By Lemma 6.2.10, a fibration structure on $\mathsf{ty}_{\mathsf{Fib}}$ is a section $U_{\mathsf{Fib}}^{\mathbb{I}} \to U_{\mathsf{Fib}}^{\mathbb{I}}.\mathsf{Fill}(\pi_1\circ-)$ over $U_{\mathsf{Fib}}^{\mathbb{I}}$, where π_1 is the small map classifier of $\mathsf{ty}_{\mathsf{Fib}}$ in (6.14), and such a section is given by a map f in the following diagram:

$$
\begin{tikzcd}
U_{\mathsf{Fib}}^{\mathbb{I}} \arrow[dr, dashed, "\langle \mathsf{id}, f\rangle"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathsf{id}"'] & & \\
& U_{\mathsf{Fib}}^{\mathbb{I}}.\mathsf{Fill}(\pi_1\circ -) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\mathsf{ty}"] \\
& U_{\mathsf{Fib}}^{\mathbb{I}} \arrow[r, "\mathsf{Fill}(\pi_1\circ -)"'] & U.
\end{tikzcd}
\tag{6.16}
$$

To find such an f, take the adjoint transpose of (6.13) to get the following commutative diagram:

$$
\begin{tikzcd}
U_{\mathsf{Fib}}^{\mathbb{I}} \arrow[r, "\overline{\pi_2}"] \arrow[d, "\pi_1^{\mathbb{I}}"'] & E \arrow[d, "\mathsf{ty}"] \\
U^{\mathbb{I}} \arrow[r, "\mathsf{Fill}(\mathsf{id}\circ -)"'] & U.
\end{tikzcd}
\tag{6.17}
$$

In the above diagram, the filling structure on $\mathsf{id} : U \to U$ is reindexed along π_1 to get a filling structure on $\mathsf{id}[\pi_1] = \pi_1$. By the reindexing lemma (6.2.11), $\mathsf{Fill}(\pi_1\circ -) = \mathsf{Fill}(\mathsf{id}\circ -)\circ\pi_1^{\mathbb{I}}$, and so the outer square in (6.16) commutes for $f = \overline{\pi_2}$. □

The following proof is a synthesis of [AGH24, Theorem 7.10] and [LOPS18, Theorem 5.2].

6.2.16. THEOREM (universal uniform fibration). *The map* $\mathsf{ty}_{\mathsf{Fib}} : E_{\mathsf{Fib}} \to U_{\mathsf{Fib}}$ *is a universal uniform fibration in the following sense:*

- *(i) every small uniform fibration* $p : A \to X$ *is a pullback of* $\mathsf{ty}_{\mathsf{Fib}} : E_{\mathsf{Fib}} \to U_{\mathsf{Fib}}$ *along a classifying map* $X \to U_{\mathsf{Fib}}$*,*
- *(ii) for every map* $\gamma : X \to U_{\mathsf{Fib}}$*, there is a fibration structure over* X*, and the associated classifying map is equal to* γ*.*

Proof. Let p be classified as a small map by $\alpha : X \to U$ and suppose p has a uniform fibration structure given by a section $f : X^{\mathbb{I}} \to X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ x)$ over $X^{\mathbb{I}}$, determined by a map $f' : X^{\mathbb{I}} \to E$ making (6.10) commute. The fibration classifier will be a map $\chi_{(\alpha,f)} : X \to U_{\mathsf{Fib}}$ such that pullback of $\mathsf{ty}_{\mathsf{Fib}}$ along $\chi_{(\alpha,f)}$ gives a map isomorphic to p:

$$
\begin{tikzcd}
A \arrow[d, "p"] & X.\chi_{(\alpha,f)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_{\mathsf{Fib}} \arrow[d] \\
X \arrow[r, phantom, "\cong"] & X \arrow[r, "\chi_{(\alpha,f)}"'] & U_{\mathsf{Fib}}.
\end{tikzcd}
\tag{6.18}
$$

Since U_{Fib} is defined by pullback in (6.13), a map $X \to U_{\mathsf{Fib}}$ can be given by a pair of maps $(X \to U, X \to \sqrt{E})$ making the appropriate square commute. Taking $\alpha : X \to U$ as the first map, it remains to find a map $X \to \sqrt{E}$ such that the following commutes:

$$
\begin{tikzcd}
X \arrow[drr, bend left] \arrow[ddr, bend right, "\alpha"'] & & \\
& U_{\mathsf{Fib}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sqrt{E} \arrow[d, "\sqrt{\mathsf{ty}}"] \\
& U \arrow[r, "\widetilde{\mathsf{Fill}(\mathsf{id}\circ -)}"'] & \sqrt{U}.
\end{tikzcd}
\tag{6.19}
$$

The fact that f gives a fibration structure on p means $\mathsf{Fill}(\alpha\circ-)\circ\mathsf{id} = \mathsf{ty}\circ f$, as this is just the outer square of (6.10). Applying Lemma 6.2.11 to $\mathsf{id} : U \to U$ and $\alpha : X \to U$, we have

$$\mathsf{Fill}(\alpha\circ-)\circ\mathsf{id} = \mathsf{Fill}(\alpha\circ-) = \mathsf{Fill}(\mathsf{id}\circ-)\circ\alpha^{\mathbb{I}},$$

and so the following square commutes:

$$\begin{tikzcd} X^{\mathbb{I}} \arrow[r, "f"] \arrow[d, "\alpha^{\mathbb{I}}"'] & E \arrow[d, "\mathsf{ty}"] \\ U^{\mathbb{I}} \arrow[r, "\mathsf{Fill}(\mathsf{id}\circ-)"'] & U. \end{tikzcd} \tag{6.20}$$

But this means (6.19) commutes for $\widetilde{f} : X \to \sqrt{E}$ the transpose of f across the $(-)^{\mathbb{I}} \dashv \sqrt{(-)}$ adjunction, since this makes the outer square the adjoint transpose of (6.20). Therefore, there is an induced map

$$\chi_{(\alpha,f)} := \langle \alpha, \widetilde{f} \rangle : X \to U_{\mathsf{Fib}}, \tag{6.21}$$

as required. Pullback of $\mathsf{ty}_{\mathsf{Fib}}$ along $\chi_{(\alpha,f)}$ gives a map isomorphic to p, by the pasting law for pullbacks and the observation that $\pi_1\circ\chi_{(\alpha,f)} = \alpha$:

$$\begin{tikzcd} X.\chi_{(\alpha,f)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_{\mathsf{Fib}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\mathsf{ty}"] \\ X \arrow[r, "\chi_{(\alpha,f)}"] \arrow[rr, bend right, "\alpha"'] & U_{\mathsf{Fib}} \arrow[r, "\pi_1"] & U. \end{tikzcd} \tag{6.22}$$

For (ii), consider any map $\gamma : X \to U_{\mathsf{Fib}}$. Then we have the following pasting of pullback diagrams

$$\begin{tikzcd} X.\pi_1[\gamma] \arrow[r] \arrow[d, "p_{\pi_1[\gamma]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E_{\mathsf{Fib}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ X \arrow[r, "\gamma"'] & U_{\mathsf{Fib}} \arrow[r, "\pi_1"'] & U, \end{tikzcd} \tag{6.23}$$

and we wish to recover the fibration structure on the type $\pi_1[\gamma]$ and show that its classifying map is γ. By Lemma 6.2.12, the fibration structure can be given by reindexing the universal fibration (Proposition 6.2.15) on $\pi_1 : U_{\mathsf{Fib}} \to U$ along γ, as in

$$\begin{tikzcd} X^{\mathbb{I}}.\mathsf{Fill}(\pi_1[\gamma]\circ-) \arrow[r] \arrow[d, shift left] \arrow[dr, phantom, "\lrcorner", very near start] & U^{\mathbb{I}}_{\mathsf{Fib}}.\mathsf{Fill}(\pi_1\circ-) \arrow[r] \arrow[d, shift left] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ X^{\mathbb{I}} \arrow[r, "\gamma^{\mathbb{I}}"'] \arrow[u, bend left, "\langle \mathsf{id}, \overline{\pi_2}\circ\gamma^{\mathbb{I}} \rangle"] & U^{\mathbb{I}}_{\mathsf{Fib}} \arrow[r, "\mathsf{Fill}(\pi_1\circ-)"'] \arrow[u, bend left, "\langle \mathsf{id}, \overline{\pi_2} \rangle"] & U \end{tikzcd}$$

The classifier of this fibration is given by (6.21)

$$\chi_{(\pi_1[\gamma], \overline{\pi_2}\circ\gamma^{\mathbb{I}})} := \langle \pi_1[\gamma], \widetilde{\overline{\pi_2}\circ\gamma^{\mathbb{I}}} \rangle$$

from which we get that the unique map $X \to U_{\mathsf{Fib}}$ making the following diagram commute is $\gamma : X \to U_{\mathsf{Fib}}$:

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left, "\widetilde{\overline{\pi_2}\circ\gamma^{\mathrm{I}}}"] \arrow[ddr, bend right, "\pi_1[\gamma]"'] & & \\
& U_{\mathsf{Fib}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sqrt{E} \arrow[d, "\sqrt{\mathsf{ty}}"] \\
& U \arrow[r, "\widetilde{\mathsf{Fill}(\mathsf{id}\circ-)}"'] & \sqrt{U}.
\end{tikzcd} \tag{6.24}$$

Commutativity of the top triangle follows from a simple calculation once the adjoint transposes are expressed as composites with the counit $\varepsilon : (\sqrt{(-)})^{\mathrm{I}} \Rightarrow \mathrm{id}_{\widehat{\mathbb{C}}}$ of the $(-)^{\mathrm{I}} \dashv \sqrt{(-)}$ adjunction. Specifically,

$$\begin{aligned}
\overline{\pi_2}\circ\gamma^{\mathrm{I}} &= (\varepsilon_E\circ\pi_2^{\mathrm{I}})\circ\gamma^{\mathrm{I}} \\
&= \varepsilon_E\circ(\pi_2\circ\gamma)^{\mathrm{I}} \\
&= \overline{(\pi_2\circ\gamma)},
\end{aligned}$$

and transposing again, as in $\widetilde{\overline{(\pi_2\circ\gamma)}}$, gives the original morphism. Therefore, $\widetilde{\overline{\pi_2}\circ\gamma^{\mathrm{I}}} = \pi_2\circ\gamma$. □

6.3 UNIFORM FIBRATIONS IN THE INTERNAL TYPE THEORY

We now turn to the problem of relating the diagrammatic reasoning in Section 6.2 to reasoning in an internal type theory. First, we show that the relevant crisp type-theoretic axioms for the tinyness of the interval from [LOPS18, Figure 1] are valid in the internal type theory $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ from Section 4.4. This is the subject of Proposition 6.3.3 and 6.3.5. We then use these axioms to show how part of the construction of the universal uniform fibration can be carried out in $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ (Proposition 6.3.8). Both of these new contributions—presenting the crisp type-theoretic axioms in an internal language and precisely relating part of the diagrammatic and type-theoretic constructions of the universal uniform fibration—have been enabled by the developments of this thesis.

Interpreting axioms for tinyness of the interval in crisp type theory

The axioms for the tinyness of the interval in [LOPS18, Figure 1] specify a type constructor $\sqrt{\ }$ with properties that correspond to being right adjoint to exponentiation by the interval. For our purposes we require the axioms corresponding to adjoint transposition, which we write in $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ as

$$\cdot \mid \cdot \vdash R : \prod\nolimits_{f :: \alpha^{\mathrm{I}} \Rightarrow \beta} \alpha \Rightarrow \sqrt{\beta} \tag{6.25}$$

$$\cdot \mid \cdot \vdash L : \prod\nolimits_{g :: \alpha \Rightarrow \sqrt{\beta}} \alpha^{\mathrm{I}} \Rightarrow \beta. \tag{6.26}$$

We show that these judgements are valid in $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ in Proposition 6.3.3. We will also need the functorial action of $\sqrt{\ }$, which is a consequence of the full axiomatisation in [LOPS18, Figure 1]:

$$\cdot \mid \cdot \vdash \sqrt{\ } : \prod\nolimits_{f :: \alpha \Rightarrow \beta} \sqrt{\alpha} \Rightarrow \sqrt{\beta}. \tag{6.27}$$

This is the subject of Proposition 6.3.5.

6.3.1. *Notation.* Using the notation introduced for the language $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ in Definition 4.4.1, we should denote the empty context in judgements in (6.25), (6.26) and (6.27) by $1 \vdash_1$ instead of $\cdot \mid \cdot$. We adopt the latter notation for the remainder of this section as it is more readable and appropriate now that we are considering genuine syntactic contexts.

To validate these axioms, we recall the fibred nature of our model of dual-context type theory admitted by a presheaf category $\widehat{\mathbb{C}}$ (Theorem 4.1.3). We will use a mixture of reasoning in the internal language of the total category $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$, detailed in Section 4.4, and reasoning in the standard internal type theory of a presheaf category. This is enabled by Remarks 4.4.19 and 4.4.20, which together say that when either the crisp part of the context is empty or the regular part of the context is empty, $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ simplifies to the standard internal type theory of a presheaf category.

We first prove a lemma specifying when a crisp variable corresponds to a global section.

6.3.2. LEMMA. *Let α and β be closed types. The judgement*

$$x :: \alpha \mid \cdot \vdash t_x : \beta[p_\alpha] \tag{6.28}$$

is valid in the internal type theory of $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ if and only if the rule

$$\frac{\cdot \mid \cdot \vdash a : \alpha}{\cdot \mid \cdot \vdash t_a : \beta} \tag{6.29}$$

is valid.

Proof. Noting that α and β are both closed types, the type $\beta[p_\alpha]$ of term t_x in judgement (6.28) is β in the following weakened context (see Notation 4.4.13):

$$\begin{tikzcd}
1.\flat\alpha \arrow[r, "p_\alpha"] \arrow[d, equal] & 1 \arrow[r, "\beta"] \arrow[d, equal] & U \arrow[d] \\
1.\flat\alpha \arrow[r, "p_\alpha"] & 1 \arrow[r, equal] & 1.
\end{tikzcd}$$

The entirety of judgement (6.28) is interpreted

$$\begin{tikzcd}
 & & & E \arrow[dl] \arrow[dd] \\
1.\flat\alpha \arrow[urrr, bend left, "t"] \arrow[rr, "{\beta[p_\alpha]}"'] \arrow[dd, equal] & & U \arrow[dd] \arrow[dr] & \\
 & & & 1 \arrow[dl] \\
1.\flat\alpha \arrow[urrr, bend left] \arrow[rr] & & 1 &
\end{tikzcd}$$

or equivalently, by Remark 4.4.7, as a section

$$\begin{tikzcd}
1.\flat\alpha.\beta[p_\alpha] \arrow[d] \\
1.\flat\alpha. \arrow[u, bend left, "{\langle \mathrm{id}, t \rangle}"]
\end{tikzcd}$$

The object $1.\flat\alpha.\beta[p_\alpha]$ is formed by the pullback

$$\begin{tikzcd}
1.\flat\alpha.\beta[p_\alpha] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1.\beta \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\
1.\flat\alpha \arrow[r, "p_\alpha"'] & 1 \arrow[r, "\beta"'] & U,
\end{tikzcd}$$

and so maps $1.\flat\alpha \to 1.\flat\alpha.\beta[p_\alpha]$ are induced by maps $s : 1.\flat\alpha \to 1.\beta$. Recalling that $1.\flat\alpha \cong \flat(1.\alpha)$, and that $\flat = p^* \circ p_*$, where p_* takes a presheaf to its set of global elements and p^* takes a set to its constant presheaf, we are interested in maps

$$p^* p_*(1.\alpha) \xrightarrow{s} 1.\beta.$$

Transposing s across the $p^* \dashv p_*$ adjunction, these correspond to maps

$$p_*(1.\alpha) \xrightarrow{\tilde{s}} p_*(1.\beta)$$

taking global elements of $1.\alpha$ to global elements of $1.\beta$. But global elements of $1.\alpha$ are the same as sections of the display map $p_\alpha : 1.\alpha \to 1$,

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[dr, equal] & 1.\alpha \arrow[d, "p_\alpha"] \\
& 1,
\end{tikzcd}$$

which is the interpretation of the premise of rule (6.29), using Proposition 4.4.19. Similarly, a global element of $1.\beta$ is a section of the display map $p_\beta : 1.\beta \to 1$, so we have

$$\begin{tikzcd}
1 \arrow[r, "\tilde{s}(a)"] \arrow[dr, equal] & 1.\beta \arrow[d, "p_\beta"] \\
& 1,
\end{tikzcd}$$

from which we obtain the interpretation of the conclusion of the rule in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$:

$$\begin{tikzcd}
& & 1.\beta \arrow[dl, "p_\beta"] \arrow[dd] \\
1 \arrow[r, equal] \arrow[urr, bend left, "\tilde{s}(a)"] \arrow[dd, equal] & 1 \arrow[dd, equal] \arrow[dr] & \\
& & 1 \arrow[dl] \\
1 \arrow[r, equal] \arrow[urr, bend left] & 1 &
\end{tikzcd}$$

□

6.3.3. PROPOSITION. *The following judgements are valid in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$:*

(i) $f :: \alpha^{\mathrm{I}} \Rightarrow \beta \mid \cdot \vdash R(f) : \alpha \Rightarrow \sqrt{\beta}$

(ii) $g :: \alpha \Rightarrow \sqrt{\beta} \mid \cdot \vdash L(g) : \alpha^{\mathrm{I}} \Rightarrow \beta$

Proof. By Lemma 6.3.2, we can prove (i) is valid by proving the validity of the rule

$$\frac{\cdot \mid \cdot \vdash f : \alpha^{\mathrm{I}} \Rightarrow \beta}{\cdot \mid \cdot \vdash R(f) : \alpha \Rightarrow \sqrt{\beta}}.$$

To interpret the type α^{I}, we apply $(-)^{\mathrm{I}}$ to the display map $p_\alpha : 1.\alpha \to 1$ and observe that $(-)^{\mathrm{I}}$ preserves the smallness of a map. Since $(p_\alpha)^{\mathrm{I}}$ is small, it must be classified by a canonical map that we call α^{I}, as in

$$\begin{tikzcd} (1.\alpha)^{\mathrm{I}} \arrow[d, "(p_\alpha)^{\mathrm{I}}"] & 1.\alpha^{\mathrm{I}} \arrow[r] \arrow[d, "p_{(\alpha^{\mathrm{I}})}"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ 1 \arrow[r, phantom, "\cong"] & 1 \arrow[r, "\alpha^{\mathrm{I}}"'] & U. \end{tikzcd}$$

Note that the codomain of $(p_\alpha)^{\mathrm{I}}$ is 1 since $(-)^{\mathrm{I}}$ is a right adjoint and so preserves the terminal object. To interpret the type $\alpha^{\mathrm{I}} \Rightarrow \beta$, we take the the exponential of the display maps $p_{(\alpha^{\mathrm{I}})} : 1.\alpha^{\mathrm{I}} \to 1$ and $p_\beta : 1.\beta \to 1$, viewed as objects in the slice category $\widehat{\mathbb{C}}/1$. We observe that the resulting object is a small map in $\widehat{\mathbb{C}}$, and so must be classified by a canonical map that we call $\alpha^{\mathrm{I}} \Rightarrow \beta$:

$$\begin{tikzcd} 1.\alpha^{\mathrm{I}} \Rightarrow 1.\beta \arrow[d, "p_{(\alpha^{\mathrm{I}})} \Rightarrow p_\beta"] & 1.(\alpha^{\mathrm{I}} \Rightarrow \beta) \arrow[r] \arrow[d, "p_{\alpha^{\mathrm{I}} \Rightarrow \beta}"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ 1 \arrow[r, phantom, "\cong"] & 1 \arrow[r, "\alpha^{\mathrm{I}} \Rightarrow \beta"'] & U. \end{tikzcd}$$

Interpretations of the types $\sqrt{\beta}$ and $\alpha \Rightarrow \sqrt{\beta}$ are given in a similar way.

The premise of the rule is a term of type $\alpha^{\mathrm{I}} \Rightarrow \beta$, interpreted by a section of the display map $p_{(\alpha^{\mathrm{I}} \Rightarrow \beta)}$, for which we can give a section f of an isomorphic map:

$$\begin{tikzcd} 1 \arrow[r, "f"] \arrow[dr, equal] & (1.\alpha)^{\mathrm{I}} \Rightarrow (1.\beta) \arrow[d, "(p_\alpha)^{\mathrm{I}} \Rightarrow p_\beta"] \\ & 1. \end{tikzcd}$$

Transposing f across the adjunction $(-) \times (1.\alpha)^{\mathrm{I}} \dashv (1.\alpha)^{\mathrm{I}} \Rightarrow (-)$ gives a map making the following commute

$$\begin{tikzcd} (1.\alpha)^{\mathrm{I}} \arrow[r, "\tilde{f}"] \arrow[dr] & 1.\beta \arrow[d, "p_\beta"] \\ & 1, \end{tikzcd}$$

then transposing $\tilde{f}$ over the adjunction $(-)^{\mathrm{I}} \dashv \sqrt{(-)}$ and back over the product-exponent adjunction $(-) \times 1.\alpha \dashv 1.\alpha \Rightarrow (-)$, we have

$$\begin{tikzcd} 1 \arrow[r, "R(f)"] \arrow[dr, equal] & (1.\alpha) \Rightarrow \sqrt{1.\beta} \arrow[d, "p_\alpha \Rightarrow \sqrt{p_\beta}"] \\ & 1, \end{tikzcd}$$

which is the interpretation of a term $R(f)$ of type $\alpha \Rightarrow \sqrt{\beta}$.

The same kind of argument can be made for the validity of judgement (ii). $\square$

6.3.4. COROLLARY. *The following judgements are valid in* $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$*:*

$$\cdot \mid \cdot \vdash R : \prod_{f :: \alpha^{\mathrm{I}} \Rightarrow \beta} \alpha \Rightarrow \sqrt{\beta}$$
$$\cdot \mid \cdot \vdash L : \prod_{g :: \alpha \Rightarrow \sqrt{\beta}} \alpha^{\mathrm{I}} \Rightarrow \beta$$

Proof. These judgements follow from applying the introduction rule for crisp Π-types (Proposition 4.4.16) to the valid judgements in Proposition 6.3.3. □

The functorial action of $\sqrt{(-)}$, given as a judgement in (6.27), can be shown to be valid using a similar style of argument to Proposition 6.3.3. It requires the full axiomatisation of the adjointness of $\sqrt{(-)}$, however, so we record it here without proof.

6.3.5. PROPOSITION. *Let* α *and* β *be crisp types. The judgement*

$$f :: \alpha \Rightarrow \beta \mid \cdot \vdash \sqrt{f} : \sqrt{\alpha} \Rightarrow \sqrt{\beta}$$

is valid in $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$.

Before moving on to the internal type-theoretic constuction of the universal uniform fibration, we make a remark about the correspondence between context morphisms and terms of function types, to be used in Theorem 6.3.8.

6.3.6. *Remark.* Recall that context morphisms act on types and elements via the operation of substitution (Notation 4.4.11). Context morphisms may be regarded as terms; we illustrate this using the internal language of a presheaf category $\widehat{\mathbb{C}}$, detailed in [AGH24, Section 3], in which a context is an object Δ in $\widehat{\mathbb{C}}$ and a context morphism is a map $f : \Delta' \to \Delta$ in $\widehat{\mathbb{C}}$.

Using the product-exponent adjunction, a context morphism $f : \Delta' \to \Delta$ is in bijective correspondence with a map $f' : 1 \to (\Delta' \Rightarrow \Delta)$. There is a map from the exponential object $\Delta' \Rightarrow \Delta$ to the terminal object that is small and so is classified, meaning we have the following pullback square and isomorphism between the display map and the original map:

$$\begin{tikzcd}
\Delta' \Rightarrow \Delta \arrow[d] & & 1.(\Delta' \Rightarrow \Delta) \arrow[r] \arrow[d, "p_{(\Delta' \Rightarrow \Delta)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\
1 & \cong & 1 \arrow[r, "\Delta' \Rightarrow \Delta"'] & U.
\end{tikzcd}$$

By the universal property of the terminal object, the map $f' : 1 \to (\Delta' \Rightarrow \Delta)$ is a section of the unique map from $\Delta' \Rightarrow \Delta$ to the terminal object, so there is a section of the display map $p_{(\Delta' \Rightarrow \Delta)}$. Since sections of display maps are terms in the internal type theory, we have the following bijective correspondence:

(i) substitutions $f : \Delta' \to \Delta$

(ii) terms of the corresponding function type, as in the judgement

$$\cdot \vdash f : \Delta' \Rightarrow \Delta.$$

Universal uniform fibration

We can now return to the goal of precisely relating internal and diagrammatic constructions of the universe of uniform fibrations, both of which rely on the right adjoint to exponentiation by the interval given by the square root functor $\sqrt{(-)}$ from Example 6.1.14.

6.3.7. *Remark.* The internal construction in [LOPS18] takes place in a type theory with a hierarchy of type universes, ensuring that the functor $\sqrt{(-)}$ can be applied to the universe U itself. As we have not set up such a hierarchy in $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$, we can only apply $\sqrt{(-)}$ to types in U. While this prevents us from immediately recovering the development of [LOPS18] in $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$, we note that it is possible to adapt $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ to include a type hierarchy. In its absence, we illustrate how to relate the internal and diagrammatic constructions with the object $\mathsf{Fib}^*(\alpha)$ from Observation 6.2.14. This is obtained by reindexing the fibration structure on a type α to get a family over X, rather than over $X^{\mathbb{I}}$, as in the diagram from (6.15):

$$\begin{tikzcd}
\mathsf{Fib}^*(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \sqrt{X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ-)} \arrow[d] \\
X \arrow[r, "\eta"'] & \sqrt{X^{\mathbb{I}}}.
\end{tikzcd}$$

We begin by relating the construction of $\mathsf{Fib}^*(\alpha)$ to the internal type theory $\mathcal{T}_{\widehat{\mathbb{C}}}$, to the extent that it is possible to do so. Let $\alpha : X \to U$ in $\widehat{\mathbb{C}}$. Suppose there is a fibration structure on the display map $p_\alpha : X.\alpha \to X$ associated with α. This means there is a filling structure on α, that is, a map $\mathsf{Fill}(\alpha\circ-) : X^{\mathbb{I}} \to U$ and, by Lemma 6.2.10, that there is a section

$$\begin{tikzcd}
X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ-) \arrow[d, "p", shift left] \\
X^{\mathbb{I}}. \arrow[u, "f", bend left]
\end{tikzcd} \tag{6.30}$$

By [AGH24, Definition 3.1], we have the following correspondence between arrows in $\widehat{\mathbb{C}}$ on the left and judgements in the internal language $\mathcal{T}_{\widehat{\mathbb{C}}}$ on the right:

a map $\alpha : X \to U$	$\leftrightsquigarrow$	a type judgement $x : X \vdash \alpha(x) : U$
a map $\mathsf{Fill}(\alpha\circ-) : X^{\mathbb{I}} \to U$	$\leftrightsquigarrow$	a type judgement $x : X^{\mathbb{I}} \vdash \mathsf{Fill}(\alpha\circ x) : U$
a section $f : X^{\mathbb{I}} \to X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ-)$ as in (6.30)	$\leftrightsquigarrow$	a term judgement $x : X^{\mathbb{I}} \vdash f : \mathsf{Fill}(\alpha\circ x)$.

The next part of the construction in $\widehat{\mathbb{C}}$ involves applying the root functor to $p : X^{\mathbb{I}}.\mathsf{Fill}(\alpha\circ-) \to X^{\mathbb{I}}$. This has no counterpart in the internal language $\mathcal{T}_{\widehat{\mathbb{C}}}$ because the $(-)^{\mathbb{I}} \dashv \sqrt{(-)}$ adjunction does not internalise (Observation 6.1.15). To overcome this problem, we switch to the internal language of $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$.

6.3.8. THEOREM. *The object* $\mathsf{Fib}^*(\alpha)$ *from Observation 6.2.14 can be defined in the internal type theory of* $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$ *as*

$$z : \textstyle\sum_{x:X}\sum_{y:\sqrt{X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ x)}} R(\mathsf{id}')(x) =_{\sqrt{X^{\mathbb{I}}}} \sqrt{p}(y) \mid \cdot \vdash$$

Proof. Let $\alpha : \mathsf{id}_X \to \mathcal{U}$ in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, that is,

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, equal] & U \arrow[d, "\mathcal{U}"] \\ X \arrow[r] & 1. \end{tikzcd}$$

This corresponds to the judgement

$$x :: X \,|\, \cdot \vdash \alpha(x) : \mathcal{U}$$

in $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$. Using the filling structure map $\mathsf{Fill}(\alpha\circ-)$ in $\widehat{\mathbb{C}}$ from Definition 6.2.2 and the universal property of the terminal object, we have the following commutative square in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$,

$$\begin{tikzcd} X^{\mathbb{I}} \arrow[r, "\mathsf{Fill}(\alpha\circ-)"] \arrow[d, equal] & U \arrow[d, "\mathcal{U}"] \\ X^{\mathbb{I}} \arrow[r] & 1, \end{tikzcd} \tag{6.31}$$

which corresponds to the judgement

$$x :: X^{\mathbb{I}} \,|\, \cdot \vdash \mathsf{Fill}(\alpha\circ x) : \mathcal{U}.$$

We will now work in the purely crisp type theory internal to the base category $\widehat{\mathbb{C}}_\flat$ of the fibred model from Theorem 4.1.3. Applying the functor $S : \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat \to \widehat{\mathbb{C}}_\flat$ (as in (4.1)) to the $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ morphism given by the diagram in (6.31), we obtain the arrow

$$X^{\mathbb{I}} \xrightarrow{\flat\mathsf{Fill}(\alpha\circ-)} \flat U$$

in $\widehat{\mathbb{C}}_\flat$. This corresponds to the judgement

$$x : X^{\mathbb{I}} \vdash \alpha(x) : \flat U$$

in the internal type theory of the base category $\mathcal{T}_{\widehat{\mathbb{C}}}$ (Remark 4.4.20). Taking the pullback of the classifer $\flat\mathsf{ty} : \flat E \to \flat U$ in $\widehat{\mathbb{C}}_\flat$ along $\flat\mathsf{Fill}(\alpha\circ-) : X^{\mathbb{I}} \to \flat U$, we have

$$\begin{tikzcd} X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ-) \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \flat E \arrow[d, "\flat\mathsf{ty}"] \\ X^{\mathbb{I}} \arrow[r, "\flat\mathsf{Fill}(\alpha\circ-)"'] & \flat U. \end{tikzcd}$$

The arrow $p : X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ-) \to X^{\mathbb{I}}$ corresponds to a term of a function type by Remark 6.3.6, as in the judgement

$$\cdot \vdash p : X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ-) \Rightarrow X^{\mathbb{I}}.$$

The next step of the category-theoretic construction is to apply the root functor $\surd : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, which we note restricts to a functor $\surd : \widehat{\mathbb{C}}_\flat \to \widehat{\mathbb{C}}_\flat$, to $p : X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ-) \to X^{\mathbb{I}}$:

$$\begin{tikzcd} \sqrt{X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ-)} \arrow[d, "\sqrt{p}"] \\ \sqrt{X^{\mathbb{I}}}. \end{tikzcd} \tag{6.32}$$

The type-theoretic counterpart is the functorial action of $\sqrt{}$ on function types, established in Remark 6.3.5, by which we obtain the judgement

$$\cdot \vdash \sqrt{p} : \sqrt{X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ-)} \Rightarrow \sqrt{X^{\mathbb{I}}}.$$

The final step of the category-theoretic construction of $\mathsf{Fib}^*(\alpha)$ is to take the pullback of the map $\sqrt{p} : \sqrt{X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ-)} \to \sqrt{X^{\mathbb{I}}}$ along the unit of the $(-)^{\mathbb{I}} \dashv \sqrt{(-)}$ adjunction at X, as in

$$\begin{tikzcd}
\mathsf{Fib}^*(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sqrt{X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ-)} \arrow[d, "\sqrt{p}"] \\
X \arrow[r, "\eta"'] & \sqrt{X^{\mathbb{I}}}.
\end{tikzcd} \tag{6.33}$$

The map $\eta : X \to \sqrt{X^{\mathbb{I}}}$ corresponds to a judgement in the internal type theory $\mathcal{T}_{\widehat{\mathbb{C}}_\flat}$ as follows. Firstly, the identity map $\mathsf{id} : X^{\mathbb{I}} \to X^{\mathbb{I}}$ corresponds to the judgement

$$\cdot \vdash \mathsf{id}' : X^{\mathbb{I}} \Rightarrow X^{\mathbb{I}}$$

by Remark 6.3.6. By Proposition 6.3.3, we then have the valid judgement

$$\cdot \vdash R(\mathsf{id}') : X \Rightarrow \sqrt{X^{\mathbb{I}}},$$

which corresponds to the map $\eta : X \to \sqrt{X^{\mathbb{I}}}$ in $\widehat{\mathbb{C}}_\flat$. Finally, the pullback in (6.33) comes from taking standard Σ-types in $\mathcal{T}_{\widehat{\mathbb{C}}_\flat}$. That is, we form the following identity type

$$x : X, y : \sqrt{X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ x)} \vdash R(\mathsf{id}')(x) =_{\sqrt{X^{\mathbb{I}}}} \sqrt{p}(y) : \flat U,$$

and apply the Σ-type formation rule twice to obtain the judgement

$$\cdot \vdash \sum_{x:X} \sum_{y:\sqrt{X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ x)}} R(\mathsf{id}')(x) =_{\sqrt{X^{\mathbb{I}}}} \sqrt{p}(y) : \flat U.$$

To move back to the dual-context internal type theory of $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, we apply the functor $T : \widehat{\mathbb{C}}_\flat \to \widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$ from the fibred model admitted by $\widehat{\mathbb{C}}$ (Theorem 4.1.3, Diagram (4.1)). This gives the object

$$\begin{tikzcd}
\mathsf{Fib}^*(\alpha) \arrow[d, equal] \\
\mathsf{Fib}^*(\alpha)
\end{tikzcd}$$

in $\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat$, which corresponds to the judgement

$$z : \sum_{x:X} \sum_{y:\sqrt{X^{\mathbb{I}}.\flat\mathsf{Fill}(\alpha\circ x)}} R(\mathsf{id}')(x) =_{\sqrt{X^{\mathbb{I}}}} \sqrt{p}(y) \mid \cdot \vdash$$

in $\mathcal{T}_{\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat}$. □

6.3.9. *Remark.* If a hierarchy of universes is added to the category $\widehat{\mathbb{C}}$ and hence to the type theory $\mathcal{T}_{\widehat{\mathbb{C}}{\downarrow}\widehat{\mathbb{C}}_\flat}$, the construction in Theorem 6.3.8 can be extended to a full internal construction of the universal uniform fibration. We do not pursue this here.

7 Hyperdoctrine semantics for quantified modal logic

Introduction

Lawvere's hyperdoctrines [Law69] provide semantics for first-order logics that reduce to familiar algebraic semantics on the propositional level. Originally conceived for intuitionistic predicate logic, they are flexible enough to be applied to other logics and extended to higher-order systems. In the case of quantified *modal* logic, hyperdoctrine semantics are significant because the traditional Kripke semantics for propositional modal logic do not automatically extend to the first-order case: that is, there are well-motivated but incomplete extensions of otherwise Kripke-complete propositional logics [Cre95].

Hyperdoctrine semantics for a first-order version of the class of *normal* modal logics are presented in [BG07], where they are used by Ghilardi as a unifying tool for studying other non-Kripkean modal semantics. In this chapter, we extend Ghilardi's hyperdoctrine semantics for modal logic in two directions—to the weaker class of *non-normal* modal logics, and to higher-order modal logics. To this end, we introduce hyperdoctrines in Section 7.1 and present the syntax of first-order non-normal modal logic in Section 7.2. With these preliminaries in hand, we define modal hyperdoctrine semantics for this logic in Section 7.3 and prove their soundness and completeness. In Section 7.4, we connect modal hyperdoctrines—in the case of S4 modal logics—to the standard notion of intuitionstic hyperdoctrine via a translation theorem. Finally, in Section 7.5, we define higher-order modal hyperdoctrine semantics for non-normal modal logics and prove their soundness and completeness.

7.1 PRELIMINARIES: HYPERDOCTRINES

Lawvere hyperdoctrines are fibred algebras indexed by categories; the algebras represent the propositional logic and the indexing category provides a type structure. Certain further conditions are imposed to capture quantification, given in the following definition (for further reference, see [Pit00]).

7.1.1. DEFINITION. Let $\mathcal{C}$ be a category with finite products and HA be the category of Heyting algebras and Heyting algebra morphisms. A *hyperdoctrine* is a functor

$$P : \mathcal{C}^{\mathrm{op}} \to \mathsf{HA}$$

such that for any projection $\pi : X \times Y \to Y$ in $\mathcal{C}$, the image $P(\pi) : P(Y) \to P(X \times Y)$ has right and left adjoints

$$\forall_\pi : P(X \times Y) \to P(Y) \text{ and } \exists_\pi : P(X \times Y) \to P(Y)$$

that satisfy corresponding *Beck-Chevalley conditions*, that is, the following diagrams commute for any $f : Z \to Y$ in $\mathcal{C}$,

$$\begin{tikzcd} P(X \times Y) \arrow[r, "\forall_\pi"] \arrow[d, "P(id_X \times f)"'] & P(Y) \arrow[d, "P(f)"] \\ P(X \times Z) \arrow[r, "\forall_{\pi'}"'] & P(Z) \end{tikzcd} \qquad \begin{tikzcd} P(X \times Y) \arrow[r, "\exists_\pi"] \arrow[d, "P(id_X \times f)"'] & P(Y) \arrow[d, "P(f)"] \\ P(X \times Z) \arrow[r, "\exists_{\pi'}"'] & P(Z), \end{tikzcd} \tag{7.1}$$

where $\pi' : X \times Z \to Z$ is a projection.

7.1.2. *Remarks.* (i) For an intuitionistic hyperdoctrine, the left adjoints $\exists_\pi : P(X \times Y) \to P(Y)$ must also satisfy the *Frobenius reciprocity condition*, omitted here as we are only concerned with classical logic, in which the quantifiers are interdefined.

(ii) Note that left and right adjoints to reindexing functors are not required to be Heyting algebra morphisms, rather any monotone maps.

The indexing category $\mathcal{C}$ represents a type structure that acts as a domain of reasoning for the logic. In this way, hyperdoctrines adopt the view that "a logic is always a logic over a type theory" [Jac99]. This is more natural from a category-theoretic perspective and subsumes untyped logics via reduction to a single type.

The restrictions placed on the syntax of our logic, if we wish to equip it with hyperdoctrine semantics, are as follows. The syntax is a typed version, built on top of a type signature and term calculus, detailed in Section 7.2. The functoriality of P means that substitution commutes with all of the logical connectives. This is clear when we consider the syntactic hyperdoctrine in Section 7.3, where we see that in order for the image of a map in the base category to be an algebra homomorphism, it is necessary that substitution commutes with the propositional connectives. Considering the syntactic hyperdoctrine also demonstrates that the Beck-Chevalley condition corresponds logically to the quantifiers commuting with substitution, and so we also require this of our syntax.

7.2 Preliminaries: typed first-order non-normal modal logic

Non-normal modal logics are a particularly weak class of modal logics, namely those that do not satisfy all the axioms of the minimal normal modal logic K. They have been of mathematical interest since the early work of Kripke, and have found applications in areas such as deontic and epistemic logic, as well as reasoning about games. In this section, we present a typed version of first-order non-normal modal logics, following [AP06] for the logic and [Pit00] for the typing. The resulting system is essentially a multi-sorted, non-normal version of the single-sorted, normal logic in [BG07].

Term calculus

The logic is built on a typed (many-sorted) *signature* Σ, consisting of *type symbols* σ, *function symbols* $F : \sigma_1, \dots, \sigma_n \to \tau$ and *relation (predicate) symbols* $R \subseteq \sigma_1, \dots, \sigma_n$. For each type σ there are *variables* $x, y, z, \dots$, and the formal expression $x : \sigma$ is a *type judgement* expressing that x is a variable of type σ. A *context* is a finite list of type judgements $x_1 : \sigma_1, \dots, x_n : \sigma_n$, denoted by Γ.

On top of the signature is a *term calculus*. The basic term calculus consists of *terms-in-context*, which are judgements $M : \sigma\ [\Gamma]$, expressing that M is a well-formed term of type σ in context Γ. The well-formed terms-in-context in the basic term calculus are inductively generated by the following rules:

- $x : \sigma\ [\Gamma, x : \sigma, \Gamma']$ is a term
- if $F : \sigma_1, \dots, \sigma_n \to \tau$ is a function symbol and $M_1 : \sigma_1\ [\Gamma], \dots, M_n : \sigma_n\ [\Gamma]$ (abbreviated $\vec{M} : \vec{\sigma}$) are terms, then $F(M_1, \dots, M_n) : \tau\ [\Gamma]$ is a term.

The meta-theoretic operation of *substitution over a term* of a term for a variable is defined by induction on the structure of an untyped term N:

- if $N = x_i$ then $N[\vec{M}/\vec{x}] = M_i$
- if $N = F(N_1, \dots, N_n)$ then $N[\vec{M}/\vec{x}] = F(N_1[\vec{M}/\vec{x}], \dots, N_n[\vec{M}/\vec{x}])$.

A *formula-in-context* is a judgement $\phi\ [\Gamma]$ expressing that ϕ is a well-formed formula in context Γ. For each relation symbol $R \subseteq \sigma_1, \dots, \sigma_n$, if $M_1 : \sigma_1\ [\Gamma], \dots, M_n : \sigma_n[\Gamma]$ are terms, then $R(M_1, \dots, M_n)\ [\Gamma]$ is an *atomic formula*. *Compound formulae* are built from the atomic formulae and the constant $\perp$ with the rules:

- $\perp\ [\Gamma]$ is a formula
- if $\phi\ [\Gamma]$ and $\psi\ [\Gamma]$ are formulae then $\phi \supset \psi\ [\Gamma]$ is a formula
- if $\phi\ [x : \sigma, \Gamma]$ is a formula then $\forall x \phi\ [\Gamma]$ is a formula
- if $\phi\ [\Gamma]$ is a formula then $\Box\phi\ [\Gamma]$ is a formula.

The remaining connectives are treated as abbreviations in the usual manner, such as equivalence of formulae $\phi \supset\subset \psi$ being an abbreviation for $\phi \supset \psi \wedge \psi \subset \phi$.

If $\phi\ [\Gamma]$ is a formula with $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$ and $M_1 : \sigma_1\ [\Gamma'], \dots, M_n : \sigma_n\ [\Gamma']$ are terms, we want to define a formula $\phi[\vec{M}/\vec{x}]\ [\Gamma']$, where every instance of the variable x_i is replaced by the term M_i, for every i. Since every formula is built in a unique way from atomic subformulae and the rules for forming compound formulae, substitution into a formula is defined on these subformulae as follows. Substitution over atomic formulae is given by

$$R(N_1,\dots,N_n)[\vec{M}/\vec{x}]\ [\Gamma'] := R(N_1[\vec{M}/\vec{x}],\dots,N_n[\vec{M}/\vec{x}])\ [\Gamma'],$$

and substitution on subformulae is as follows, where x_{m+1} is a fresh variable:

- $\bot[\vec{M}/\vec{x}]\ [\Gamma'] := \bot\ [\Gamma']$
- $(\phi_1 \supset \phi_2)[\vec{M}/\vec{x}]\ [\Gamma'] := (\phi_1[\vec{M}/\vec{x}]) \supset (\phi_2[\vec{M}/\vec{x}])\ [\Gamma']$
- $(\forall x_{n+1}\psi)[\vec{M}/\vec{x}]\ [\Gamma'] := \forall x_{m+1}(\psi[\vec{M}/\vec{x}, x_{m+1}/x_{n+1}])\ [\Gamma']$
- $(\Box\psi)[\vec{M}/\vec{x}]\ [\Gamma'] := \Box(\psi[\vec{M}/\vec{x}])\ [\Gamma']$.

Logical calculus

A Hilbert-style system for (typed) non-normal propositional modal logics is given by any axiomatisation of propositional logic, plus the rules and axiom schema

(RE)
$$\frac{\phi \supset\subset \psi \quad [\Gamma]}{\Box\phi \supset\subset \Box\psi \quad [\Gamma]}$$

(MP)
$$\frac{\phi \quad [\Gamma] \qquad \phi \supset \psi \quad [\Gamma]}{\psi \quad [\Gamma]}$$

(E)
$$\Diamond\phi \supset\subset \neg\Box\neg\phi \quad [\Gamma]$$

and zero or more of the following axiom schemata:

(M)
$$\Box(\phi \wedge \psi) \supset (\Box\phi \wedge \Box\psi) \quad [\Gamma]$$

(C)
$$(\Box\phi \wedge \Box\psi) \supset \Box(\phi \wedge \psi) \quad [\Gamma]$$

(N)
$$\Box\top \quad [\Gamma]$$

The smallest non-normal propositional modal logic is called E. The non-normal extensions are denoted by E_X, where X is a subset of $\{\mathsf{M}, \mathsf{C}, \mathsf{N}\}$ and E_X is the smallest system containing every instance of the axiom schemata in X. Non-normal modal logics refer to these eight systems. Note that the system EMCN is equivalent to the smallest normal modal logic K. We also note here that the logic S4 is the system K plus the following schemata:

(T)
$$\Box\phi \supset \phi \quad [\Gamma]$$

(4)
$$\Box\phi \supset \Box\Box\phi \quad [\Gamma]$$

While we restrict ourselves to non-normal modal logics for the present, we will be concerned with S4 in Section 7.4.

To extend any propositional non-normal modal logic S to a (typed) first-order logic TFOL + S, we add the following axiom schema and rules:[1]

($\forall$-ELIMINATION)
$$(\forall x\phi)[x_1,\dots,x_n] \supset \phi \quad [x : \sigma, \Gamma]$$

($\forall$-INTRODUCTION)
$$\frac{\phi[x_1,\dots,x_n] \supset \psi \quad [x : \sigma, \Gamma]}{\phi \supset \forall x\psi \quad [\Gamma]}$$

(INSTANTIATION)
$$\frac{\phi \quad [\Gamma]}{\phi[\vec{M}/\vec{x}] \quad [\Gamma']}$$

where $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$ and $\vec{M}$ abbreviates the terms $M_1 : \sigma_1\ [\Gamma'], \dots, M_n : \sigma_n\ [\Gamma']$. The formula $\phi[x_1,\dots,x_n]$ evidentiates the free variables of ϕ, since the rule requires that x is not free in ϕ. It differs from the formula $\phi\ [\Gamma]$ by the renaming of bound variables.

[1]This axiomatisation deviates from [AP06], instead following [BG07] in taking two separate principles of *replacement*—corresponding to the *Instantiation* rule—and *agreement*—corresponding to the $\forall$-*Introduction* rule—to more readily accommodate the proofs.

A *derivation* of a formula $\phi\ [\Gamma]$ is a finite sequence of formulae $\phi_1\ [\Gamma_1], \phi_2\ [\Gamma_2], \dots, \phi_n\ [\Gamma_n]$ such that each formula is either an instance of an axiom schema or follows from earlier formulae by one of the rules of inference. A formula $\phi\ [\Gamma]$ is said to be *derivable* in the axiom system $\mathsf{TFOL} + \mathsf{S}$ if there exists a derivation of $\phi\ [\Gamma]$ in this axiom system, denoted $\vdash_{\mathsf{TFOL+S}} \phi\ [\Gamma]$.

7.3 HYPERDOCTRINE SEMANTICS FOR $\mathsf{TFOL} + \mathsf{E}_X$

Before defining a modal hyperdoctrine, we present the standard algebraic semantics for modal logic, to which the hyperdoctrine semantics reduce on the propositional level. Algebraic semantics for modal logic S4 were developed by McKinsey and Tarski, extended to normal modal logics in [Lem66], and even weaker modal logics in [Dos89]. We adopt this last, most general definition of modal algebra.

7.3.1. DEFINITION. A *modal algebra* A is a Boolean algebra $(A, \wedge_A, \vee_A, \neg_A, \top_A, \bot_A)$ together with a unary operator $\Box_A$ satisfying zero or more conditions, such as:

$$\Box_A(x \wedge_A y) \leq \Box_A(x) \wedge_A \Box_A(y) \tag{M_A}$$

$$\Box_A(x) \wedge_A \Box_A(y) \leq \Box_A(x \wedge_A y) \tag{C_A}$$

$$\Box_A(\top_A) = \top_A \tag{N_A}$$

$$\Box_A(x) \leq x \tag{T_A}$$

$$\Box_A(x) \leq \Box_A\Box_A(x) \tag{4_A}$$

There are secondary operations $x \supset_A y := \neg_A x \vee_A y$ and $\Diamond_A(x) := \neg_A\Box_A(\neg_A x)$.

We use the same notation for the operations on the algebra as for the logical connectives, to highlight their correspondence. The algebraic operations are subscripted with the underlying set when it is helpful to have a reminder that we are in the algebraic setting. A poset structure is inherited from the Boolean algebra, given by the order $x \leq y$ if and only if $x \wedge_A y = x$. Modal algebras and structure-preserving functions between them form the category MA.

Possible conditions on $\Box_A$ correspond to axiom schemata of the logical calculus to be captured. In the proofs that follow, we only specify the strength of modal algebra to which the category MA refers when necessary. Since we are concerned with the level of predicates, most proofs operate independently of the specific axioms satisfied by the modal operator.

Modal hyperdoctrine semantics

In this section, we adapt the definition of Lawvere hyperdoctrine from intuitionistic logic to modal logic, define *interpretation* in a modal hyperdoctrine, and prove that this gives sound and complete semantics for $\mathsf{TFOL} + \mathsf{E}_X$.

7.3.2. DEFINITION. Let $\mathcal{C}$ be a category with finite products. A *modal hyperdoctrine* is a contravariant functor $P : \mathcal{C}^{\mathrm{op}} \to \mathsf{MA}$ such that for any projection $\pi : X \times Y \to Y$ in $\mathcal{C}$, $P(\pi) : P(Y) \to P(X \times Y)$ has a right adjoint satisfying the Beck-Chevalley condition (7.1).

Since our modal logic is classical, our definition of modal hyperdoctrine does not treat the existential quantifier independently.

7.3.3. DEFINITION. Fix a modal hyperdoctrine $P : \mathcal{C}^{\mathrm{op}} \to \mathsf{MA}$. An *interpretation* $[\![\text{-}]\!]$ of $\mathsf{TFOL} + \mathsf{E}_X$ in P consists of the following:

- assignment of an object $[\![\sigma]\!]$ in $\mathcal{C}$ to each basic type σ in $\mathsf{TFOL} + \mathsf{E}_X$
- assignment of an arrow $[\![F]\!] : [\![\sigma_1]\!]\times\cdots\times[\![\sigma_n]\!] \to [\![\tau]\!]$ in $\mathcal{C}$ to each function symbol $F : \sigma_1, \dots, \sigma_n \to \tau$ in $\mathsf{TFOL} + \mathsf{E}_X$
- assignment of an element $[\![R\ [\Gamma]]\!]$ in the modal algebra $P([\![\Gamma]\!])$ to each typed predicate symbol $R\ [\Gamma]$ in $\mathsf{TFOL} + \mathsf{E}_X$; if the context Γ is $x_1 : \sigma_1, ..., x_n : \sigma_n$, then $[\![\Gamma]\!]$ denotes $[\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!]$.

The interpretation of a term is defined by induction on its derivation, as follows:

- $[\![x : \sigma\ [\Gamma, x : \sigma, \Gamma']]\!]$ is defined as the following projection in $\mathcal{C}$:

$$\pi : [\![\Gamma]\!] \times [\![\sigma]\!] \times [\![\Gamma']\!] \to [\![\sigma]\!]$$

- $[\![F(M_1, \dots, M_n) : \tau\ [\Gamma]]\!] := [\![F]\!]\circ\langle[\![M_1 : \sigma_1\ [\Gamma]]\!], \dots, [\![M_n : \sigma_n\ [\Gamma]]\!]\rangle$.

Formulae are interpreted inductively in the following manner:

- $[\![R(M_1, \dots, M_n)\ [\Gamma]]\!] := P(\langle[\![M_1 : \sigma_1\ [\Gamma]]\!], \dots, [\![M_n : \sigma_n\ [\Gamma]]\!]\rangle)([\![R]\!])$
- for the propositional connectives,

$$\begin{aligned}
[\![\phi \wedge \psi\ [\Gamma]]\!] &:= [\![\phi\ [\Gamma]]\!] \wedge_{P([\![\Gamma]\!])} [\![\psi\ [\Gamma]]\!] \\
[\![\phi \vee \psi\ [\Gamma]]\!] &:= [\![\phi\ [\Gamma]]\!] \vee_{P([\![\Gamma]\!])} [\![\psi\ [\Gamma]]\!] \\
[\![\phi \supset \psi\ [\Gamma]]\!] &:= [\![\phi\ [\Gamma]]\!] \supset_{P([\![\Gamma]\!])} [\![\psi\ [\Gamma]]\!] \\
[\![\neg\phi\ [\Gamma]]\!] &:= \neg_{P([\![\Gamma]\!])}[\![\phi\ [\Gamma]]\!] \\
[\![\Box\phi\ [\Gamma]]\!] &:= \Box_{P([\![\Gamma]\!])}([\![\phi\ [\Gamma]]\!]) \\
[\![\bot\ [\Gamma]]\!] &:= \bot_{P([\![\Gamma]\!])} \\
[\![\top[\Gamma]]\!] &:= \top_{P([\![\Gamma]\!])}
\end{aligned}$$

- for the quantifiers,

$$\begin{aligned}
[\![\forall x\phi\ [\Gamma]]\!] &:= \forall_\pi([\![\phi\ [x : \sigma, \Gamma]]\!]) \\
[\![\exists x\phi\ [\Gamma]]\!] &:= \exists_\pi([\![\phi\ [x : \sigma, \Gamma]]\!])
\end{aligned}$$

  where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ is a projection in $\mathcal{C}$.

For a formula $\phi\ [\Gamma]$, where $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$, and terms $M_1 : \sigma_1\ [\Gamma'], \dots, M_n : \sigma_n\ [\Gamma']$, the interpretation of substitution by $\vec{M}$ is:

$$[\![\phi[\vec{M}/\vec{x}]\ [\Gamma']]\!] = P(\langle[\![M_1 : \sigma_1\ [\Gamma']]\!], \dots, [\![M_n : \sigma_n\ [\Gamma']]\!]\rangle)([\![\phi\ [\Gamma]]\!]).$$

This can be proved by induction on the structure of ϕ. Weakening of the context of a formula $\phi\ [\Gamma]$ to the context $x : \sigma, \Gamma$ is the following special case:

$$[\![\phi\ [x : \sigma, \Gamma]]\!] = P(\pi)([\![\phi\ [\Gamma]]\!])$$

where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ is a projection map.

7.3.4. DEFINITION. A formula $\phi\ [\Gamma]$ is *satisfied* in an interpretation $[\![\text{-}]\!]$ in a modal hyperdoctrine P if and only if $[\![\phi]\!] = \top_{P([\![\Gamma]\!])}$.

Since $a \leq_A \top_A$ holds for every element a in a Boolean algebra A, showing the satisfiability of ϕ [Γ] amounts to showing $\top_{P(\llbracket\Gamma\rrbracket)} \leq \llbracket\phi\rrbracket$. Note that the definition of satisfaction here differs from that in [Pit00], which is concerned with the satisfiability of sequents rather than formulae.

Soundness and completeness

We now prove the soundness and completeness of $\mathsf{TFOL} + \mathsf{E}_X$ with respect to the modal hyperdoctrine semantics. We make use of an equivalent condition for the satisfaction of an implication $\phi \supset \psi$ [Γ] in an interpretation:

$$\top \leq \llbracket \phi \supset \psi \ [\Gamma] \rrbracket \text{ if and only if } \llbracket \phi \ [\Gamma] \rrbracket \leq \llbracket \psi \ [\Gamma] \rrbracket. \tag{7.2}$$

This follows from the fact that in a Boolean algebra, the pair of functions $- \wedge x \,:\, A \to A$ and $x \supset - \,:\, A \to A$ determine an adjunction, that is, for all $y, z \in A$, $z \leq x \supset y$ if and only if $z \wedge x \leq y$. Letting $z = \top$ and using the fact that $\top \wedge x = x$, we have $\top \leq x \supset y$ if and only if $x \leq y$.

We will also use the following bijection, coming from the adjointness condition on the universal quantifier:

$$\frac{P(\pi)(A) \leq_{P(X \times Y)} B}{A \leq_{P(Y)} \forall_\pi(B)} \tag{7.3}$$

The following soundness proof is with respect to the systems $\mathsf{TFOL} + \mathsf{E}_X$, but we note that the proof applies to other systems $\mathsf{TFOL} + \mathsf{S}$, provided we strengthen the conditions on the modal operator in correspondence with the axiom schemata of S. This generality is possible given how the predicate and propositional components interact in the semantics, that is, the structure on the predicate part governs the interaction between the modal algebras, while preserving their internal structure.

7.3.5. THEOREM. *If ϕ [Γ] has a derivation in $\mathit{TFOL} + E_X$, then it is satisfied in any interpretation in any modal hyperdoctrine satisfying the algebraic conditions corresponding to the set of axioms X.*

Proof. Fix a modal hyperdoctrine P and an interpretation $\llbracket - \rrbracket$ in P. The proof is by induction on the derivation of ϕ [Γ], which amounts to checking that all axiom schemata are satisfied and that all rules preserve satisfaction.

For the propositional fragment, beginning with rule RE, suppose $\llbracket \phi \supset\subset \psi \rrbracket$ is true in $P(\llbracket\Gamma\rrbracket)$. Expanding the abbreviation $\supset\subset$ and taking the interpretation of the connectives as in Definition 7.3.3, we have:

$$\llbracket \phi \supset \psi \wedge \psi \supset \phi \rrbracket = \llbracket\phi\rrbracket \supset \llbracket\psi\rrbracket \wedge \llbracket\psi\rrbracket \supset \llbracket\phi\rrbracket.$$

It is a theorem in a Boolean algebra that the right-hand side implies $\llbracket\phi\rrbracket = \llbracket\psi\rrbracket$. Therefore, $\Box\llbracket\phi\rrbracket = \Box\llbracket\psi\rrbracket$, which is the interpretation of the formula $\Box\phi \supset\subset \Box\psi$ [Γ], and so the rule RE preserves satisfaction. The rule MP may be checked in a similar way.

For the modal axiom schemata, satisfaction of schema E corresponds to the definition of the $\Diamond$ operator in a modal algebra, so $\Diamond\llbracket\phi\rrbracket = \neg\Box\neg\llbracket\phi\rrbracket$ in $P(\llbracket\Gamma\rrbracket)$. The interpretation of schema M is:

$$\llbracket \Box(\phi \wedge \psi) \supset (\Box\phi \wedge \Box\psi) \rrbracket = \Box(\llbracket\phi\rrbracket \wedge \llbracket\psi\rrbracket) \supset (\Box\llbracket\phi\rrbracket \wedge \Box\llbracket\psi\rrbracket),$$

so M is satisfied in the interpretation if $\Box(\llbracket\phi\rrbracket \wedge \llbracket\psi\rrbracket) \leq (\Box\llbracket\phi\rrbracket \wedge \Box\llbracket\psi\rrbracket)$, by the equivalent condition for satisfaction of an implication established in (7.2). This corresponds clearly to condition M_A on the operator.

In a similar way, we can show that schema N corresponds to condition N_A and C corresponds to C_A. It is also clear that we may add more axiom schemata to $\mathsf{TFOL} + \mathsf{E}_X$ to get a system $\mathsf{TFOL} + \mathsf{S}$, and that these schemata are satisfied in the interpretation if we add corresponding conditions on the modal operator in the algebra. Satisfaction of the axiom schemata for the non-modal part of the propostional logic may be verified in the same way.

For the first-order fragment, the axiom schema $\forall$-*Elimination* is satisfied if and only if the interpretation $\llbracket(\forall x\phi)[x_1, \dots, x_n] \supset \phi\rrbracket$ is true in $P(\llbracket\sigma\rrbracket \times \llbracket\Gamma\rrbracket)$. By (7.2), we can do this by showing

$$\llbracket(\forall x\phi)[x_1, \dots, x_n]\rrbracket \leq \llbracket\phi\rrbracket.$$

The logical expression on the left-hand side, $(\forall x\phi)[x_1, \dots, x_n]$, is a formula in context $[x : \sigma, \Gamma]$, but which does not contain x, and so corresponds to weakening of the context. By the semantics of substitution, we have:

$$\llbracket(\forall x\phi)[x_1, \dots, x_n]\ [x : \sigma, \Gamma]\rrbracket = P(\pi)(\llbracket\forall x\phi\ [\Gamma]\rrbracket),$$

and by the interpretation of the universal quantifier,

$$P(\pi)(\llbracket\forall x\phi\ [\Gamma]\rrbracket) = P(\pi)(\forall_\pi(\llbracket\phi\ [x : \sigma, \Gamma]\rrbracket)).$$

This turns the desired statement into another form of the adjointness condition for universal quantification, that is, the counit characterisation:

$$P(\pi)(\forall_\pi(\llbracket\phi\ [x : \sigma, \Gamma]\rrbracket)) \leq \llbracket\phi\ [x : \sigma, \Gamma]\rrbracket.$$

To show that the $\forall$-*Introduction* rule preserves satisfaction, suppose

$$\llbracket\phi[x_1, \dots, x_n] \supset \psi\rrbracket$$

is true in $P(\llbracket\sigma\rrbracket \times \llbracket\Gamma\rrbracket)$, or equivalently, $\llbracket\phi[x_1, \dots, x_n]\rrbracket \leq \llbracket\psi\rrbracket$. Then we need to show that $\llbracket\phi \supset \forall x\psi\rrbracket$ is true in $P(\llbracket\Gamma\rrbracket)$, or equivalently, $\llbracket\phi\rrbracket \leq \forall_\sigma\llbracket\psi\rrbracket$. This logical rule directly translates to one direction of the adjointness correspondence when we observe that the formula $\phi[x_1, \dots, x_n]\ [x : \sigma, \Gamma]$ is weakening of the formula $\phi[\Gamma]$. By the interpretation of substitution,

$$\llbracket\phi[x_1, \dots, x_n]\ [x : \sigma, \Gamma]\rrbracket = P(\pi)(\llbracket\phi\ [\Gamma]\rrbracket).$$

But if $P(\pi)(\llbracket\phi\rrbracket) \leq \llbracket\psi\rrbracket$ holds, then by the adjointness condition for universal quantification, $\llbracket\phi\rrbracket \leq \forall_\sigma\llbracket\psi\rrbracket$ as required.

For the *Instantiation* rule, suppose $\top \leq \llbracket\phi\rrbracket$ in $P(\llbracket\Gamma\rrbracket)$. Applying the (order-preserving) modal algebra homomorphism

$$P(\langle\llbracket M_1 : \sigma_1\ [\Gamma']\rrbracket, \dots, \llbracket M_n : \sigma_n\ [\Gamma']\rrbracket\rangle) : P(\llbracket\Gamma\rrbracket) \to P(\llbracket\Gamma'\rrbracket)$$

to both sides, we get

$$P(\langle [\![M_1 : \sigma_1\ [\Gamma']]\!], \dots, [\![M_n : \sigma_n\ [\Gamma']]\!]\rangle)(\top) \leq P(\langle [\![M_1 : \sigma_1\ [\Gamma']]\!], \dots, [\![M_n : \sigma_n\ [\Gamma']]\!]\rangle)([\![\phi\ [\Gamma]]\!]).$$

Since modal algebra homomorphisms preserve $\top$, and by the semantics of substitution, we have $\top \leq [\![\phi[\vec{M}/\vec{x}]]\!]$ in $P([\![\Gamma']\!])$. □

Towards proving completeness, we now define the syntactic hyperdoctrine of $\mathsf{TFOL} + \mathsf{S}$. For the base category $\mathcal{C}$, let the objects be contexts Γ up to α-equivalence (renaming of variables). This is equivalent to taking as objects lists of types $\sigma_1, \dots, \sigma_n$, rather than a list of variable-type pairs. A context morphism from $\sigma_1, \dots, \sigma_n$ to $\Gamma' = \tau_1, \dots, \tau_m$ is given by a list of terms $t_1 : \tau_1[\Gamma], \dots, t_m : \tau_m[\Gamma]$, abbreviated $[t_1, \dots, t_m] : \Gamma \to \Gamma'$. We take as arrows equivalence classes of context morphisms under the relation $[t_1, \dots, t_n] = [s_1, \dots, s_n]$ if and only if t_i is equivalent—as terms—to s_i, for all i. Contexts up to α-equivalence and context morphisms up to term-equivalence form a category.

7.3.6. DEFINITION. For a context Γ, let $\mathsf{Form}_\Gamma := \{\phi \mid \phi \text{ is a formula in context } \Gamma\}$. The *syntactic hyperdoctrine* $P : \mathcal{C}^{\mathrm{op}} \to \mathsf{MA}$ sends objects Γ to

$$P(\Gamma) := \mathsf{Form}_\Gamma / \sim$$

where $\sim$ is the equivalence relation $\phi \sim \psi$ if and only if $\vdash_{\mathsf{TFOL+S}} \phi \supset\subset \psi\ [\Gamma]$. The object $P(\Gamma)$ has a modal algebra structure induced by the logical connectives.

The syntactic hyperdoctrine sends arrows $[t_1, \dots, t_m] : \Gamma \to \Gamma'$ to

$$P([t_1, \dots, t_m]) : P(\Gamma') \to P(\Gamma),$$

defined by $P([t_1, \dots, t_m])(\phi) := \phi[t_1/y_1, \dots, t_m/y_m]$.

7.3.7. PROPOSITION. *The syntactic hyperdoctrine $P : \mathcal{C}^{op} \to MA$ is a modal hyperdoctrine.*

Proof. Firstly, the base category $\mathcal{C}$ has finite products: for $\Gamma = \sigma_1, \dots, \sigma_n$ and $\Gamma' = \tau_1, \dots, \tau_m$, define $\Gamma \times \Gamma'$ as $\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_n$. We then have, as an associated projection,

$$[y_1, \dots, y_m] : \Gamma \times \Gamma' \to \Gamma',$$

where the y_i are variables $y_i : \tau_i\ [\Gamma, \Gamma']$. The other projection is defined similarly, and it is straightforward to show that this gives a categorical product in $\mathcal{C}$.

Next, we check that the codomain of P is in fact the category of modal algebras and structure-preserving homomorphisms. For a context Γ in $\mathcal{C}$, $P(\Gamma)$ forms a modal algebra with operations induced in the expected way by the logical connectives. Considering only the non-modal fragment of the logic, $P([t_1, \dots, t_n])$ is a Boolean algebra homomorphism since substitution commutes with all the non-modal logical operations. To extend this to a modal algebra homomorphism, we require that $P([t_1, \dots, t_n])$ preserves the modal operator $\Box$ and any extra conditions placed on $\Box$. This follows from the fact that $P([t_1, \dots, t_n])$ performs substitution into a formula, and the syntax specifies that $\Box$ commutes with substitution.

Proceeding to the quantifier structure, universal quantification is given by a right adjoint to $P(\pi) : P(\Gamma') \to P(\Gamma \times \Gamma')$, where $\pi : \Gamma \times \Gamma' \to \Gamma'$ is the second projection in $\mathcal{C}$. Let ψ be a formula in

$P(\Gamma \times \Gamma')$; since the following arguments respect equivalence, we will identify ψ with the equivalence class to which it belongs. Define $\forall_\pi : P(\Gamma \times \Gamma') \to P(\Gamma')$ by

$$\forall_\pi(\psi) := \forall x_1 \dots \forall x_n \psi,$$

with the formula on the right hand side denoting the corresponding equivalence class.

Suppose $\phi \in P(\Gamma')$; to show that $\forall_\pi$ is the right adjoint of $P(\pi)$ means showing $P(\pi)(\phi) \leq \psi$ in $P(\Gamma \times \Gamma')$ if and only if $\phi \leq \forall x_1 \dots \forall x_n \psi$ in $P(\Gamma')$. For the first direction, assume $P(\pi)(\phi) \leq \psi$ in $P(\Gamma \times \Gamma')$. Since $P(\pi)(\phi)$ corresponds to weakening of the context of ϕ $[\Gamma]$ to $\phi[y_1, \dots, y_m]$ $[\Gamma, \Gamma']$, we have $\phi[y_1, \dots, y_m] \leq \psi$ in $P(\Gamma \times \Gamma')$. The partial order in $P(\Gamma \times \Gamma')$ is induced by its lattice structure, so the above ordering corresponds to the equation $\phi[y_1, \dots, y_m] \wedge \psi = \phi[y_1, \dots, y_m]$. By the definition of the syntactic hyperdoctrine, we can make the following derivability statement:

$$\vdash_{\mathsf{TFOL+S}} \phi[y_1, \dots, y_m] \wedge \psi \supset\subset \phi[y_1, \dots, y_m] \; [\Gamma, \Gamma']$$

from which it follows that

$$\vdash_{\mathsf{TFOL+S}} \phi[y_1, \dots, y_m] \supset \psi \; [\Gamma, \Gamma'].$$

Repeated application of $\forall$-*Introduction* gives

$$\vdash_{\mathsf{TFOL+S}} \phi \supset \forall x_1 \dots \forall x_n \psi \quad [\Gamma'],$$

from which it follows that

$$\vdash_{\mathsf{TFOL+S}} \phi \wedge \forall x_1 \dots \forall x_n \psi \supset\subset \phi \quad [\Gamma'].$$

Translating back to the modal algebra, this means $\phi \wedge \forall x_1 \dots \forall x_n \psi = \phi$ in $P(\Gamma')$, and so $\phi \leq \forall x_1 \dots \forall x_n \psi$ in $P(\Gamma')$, as required.

For the other direction, assume $\phi \leq \forall x_1 \dots \forall x_n \psi$ in $P(\Gamma')$. Using the same reasoning as before to translate from a statement in the modal algebra to one in the logic, we have

$$\vdash_{\mathsf{TFOL+S}} \phi \wedge \forall x_1 \dots \forall x_n \psi \supset\subset \phi \quad [\Gamma'],$$

from which it follows:

$$\vdash_{\mathsf{TFOL+S}} \phi \supset \forall x_1 \dots \forall x_n \psi \quad [\Gamma'].$$

Applying the *Instantiation* rule to weaken the context gives

$$\vdash_{\mathsf{TFOL+S}} (\phi \supset \forall x_1 \dots \forall x_n \psi)[y_1, \dots, y_m] \quad [x_1 : \sigma_1, \Gamma'],$$

where we substitute for the variables $y_i : \tau_i$ $[\Gamma']$ variables $y_i : \tau_i$ $[x_1 : \sigma_1, \Gamma']$. Since substitution commutes with $\supset$, we have

$$\vdash_{\mathsf{TFOL+S}} \phi[y_1, \dots, y_m] \supset (\forall x_1 \dots \forall x_n \psi)[y_1, \dots, y_m] \quad [x_1 : \sigma, \Gamma'] \tag{7.4}$$

We will prove $\phi[y_1, \dots, y_m] \supset \forall x_2 \dots \forall x_n \psi$ $[x_1 : \sigma_1, \Gamma']$ using the deduction theorem. Assume

$$\vdash_{\mathsf{TFOL+S}} \phi[y_1, \dots, y_m] \; [x_1 : \sigma_1, \Gamma'], \tag{7.5}$$

then applying rule MP (modus ponens) to (7.5) and (7.4) gives:

$$\vdash_{\mathsf{TFOL+S}} (\forall x_1 \ldots \forall x_n \psi)[y_1, \ldots, y_m] \quad [x_1 : \sigma_1, \Gamma']. \tag{7.6}$$

The following is an instance of the $\forall$-*Elimination* schema:

$$\vdash_{\mathsf{TFOL+S}} (\forall x_1 \forall x_2 \ldots \forall x_n \psi)[y_1, \ldots, y_m] \supset \forall x_2 \ldots \forall x_n \psi \quad [x_1 : \sigma_1, \Gamma']. \tag{7.7}$$

Applying modus ponens to (7.6) and (7.7):

$$\vdash_{\mathsf{TFOL+S}} \forall x_2 \ldots \forall x_n \psi \, [x_1 : \sigma_1, \Gamma'].$$

Since this follows from the assumption that $\phi[y_1, \ldots, y_m]\,[x_1 : \sigma_1, \Gamma']$ is derivable, we have

$$\vdash_{\mathsf{TFOL+S}} \phi[y_1, \ldots, y_m] \supset \forall x_2 \ldots \forall x_n \psi \, [x_1 : \sigma_1, \Gamma']$$

by the deduction theorem. Repeating this argument, we get

$$\vdash_{\mathsf{TFOL+S}} \phi[y_1, \ldots, y_m] \supset \psi \, [\Gamma, \Gamma'],$$

and translating this back into a statement in the modal algebra $P(\Gamma')$, we have $P(\pi)\phi \leq \psi$.

To show that the corresponding Beck-Chevalley condition is satisfied, let $\Gamma'' = v_1 : \mu_1, \ldots, v_l : \mu_l$ be a context up to α-equivalence. Then for every context morphism $[s_1, \ldots, s_m] : \Gamma'' \to \Gamma'$ with $s_i : \tau_i \, [\Gamma'']$ the following diagram must commute:

$$\begin{tikzcd}
P(\Gamma \times \Gamma') \arrow[r, "\forall_\pi"] \arrow[d, "P(\mathrm{id}_\Gamma \times [s_1,\ldots,s_m])"'] & P(\Gamma') \arrow[d, "P([s_1,\ldots,s_m])"] \\
P(\Gamma \times \Gamma'') \arrow[r, "\forall_{\pi'}"'] & P(\Gamma'')
\end{tikzcd}$$

where $\pi' : \Gamma \times \Gamma'' \to \Gamma''$ is a projection. Since we specified in the term calculus that the quantifiers commute with substitution, we can make the following argument, for $\psi \in P(\Gamma \times \Gamma')$:

$$\begin{aligned}
P([s_1, \ldots, s_m]) \circ \forall_\pi(\psi) &= P([s_1, \ldots, s_m])(\forall x_1 \ldots \forall x_n \psi) \\
&= (\forall x_1 \ldots \forall x_n \psi)[s_1/y_1, \ldots, s_m/y_m] \\
&= \forall x_1 \ldots \forall x_n (\psi[s_1/y_1, \ldots, s_m/y_m]) \\
&= \forall_{\pi'}(\psi[s_1/y_1, \ldots, s_m/y_m]) \\
&= \forall_{\pi'} \circ P(1_\Gamma \times [s_1, \ldots, s_m])(\psi)
\end{aligned}$$

□

There is the obvious canonical interpretation (generic model) of $\mathsf{TFOL} + \mathsf{S}$ in the syntactic hyperdoctrine, about which we can say the following.

7.3.8. PROPOSITION. *If $\phi\,[\Gamma]$ is satisfied in the canonical interpretation in the syntactic hyperdoctrine then it is deducible in $\mathsf{TFOL} + \mathsf{S}$.*

From this it follows that if $\phi\,[\Gamma]$ is satisfied in any interpretation in any modal hyperdoctrine, then it is deducible in $\mathsf{TFOL} + \mathsf{S}$.

7.4 Hyperdoctrinal translation theorem for $\mathrm{TFOL} + \mathrm{S4}$

Having categorical semantics for a logic allows us to investigate that logic using category theory. In the following, we compose an S4 modal hyperdoctrine with a translation functor from modal to intuitionistic logic to get a hyperdoctrinal translation theorem. One direction of the Gödel-McKinsey-Tarski translation between modal and intuitionistic logic (see, for example, [MT46]) can be expressed as the functor

$$\mathsf{Fix}_{\Box} : \mathsf{MA} \to \mathsf{HA}$$

sending a modal algebra A to a Heyting algebra on the set $\{a \in A \mid \Box a = a\}$, and an MA-homomorphism $h : A \to B$ to an HA-homomorphism

$$\mathsf{Fix}_{\Box}(h) : \mathsf{Fix}_{\Box}(A) \to \mathsf{Fix}_{\Box}(B).$$

For the functor to send modal algebras to Heyting algebras, the modal algebra must satisfy all the axioms in Definition 7.3.1, and thus the translation theorem only works for modal logics S4 and stronger.

7.4.1. Proposition. *Let $P : \mathcal{C}^{op} \to MA$ be an S4-modal hyperdoctrine, that is, a modal hyperdoctrine where MA satisfies the algebraic conditions corresponding to the logic S4. Then the functor*

$$P_{\mathsf{Fix}} := \mathsf{Fix}_{\Box} \circ P : \mathcal{C}^{op} \to HA$$

is an intuitionistic hyperdoctrine.

Proof. Firstly, we show that there are right and left adjoints, $\forall_{\pi}^{\Box}$ and $\exists_{\pi}^{\Box}$, to

$$P_{\mathsf{Fix}}(\pi) : P_{\mathsf{Fix}}(Y) \to P_{\mathsf{Fix}}(X \times Y),$$

where $\pi : X \times Y \to Y$ is a projection function in $\mathcal{C}$. Since P is a modal hyperdoctrine, there exist maps $\forall_{\pi}, \exists_{\pi} : P(X \times Y) \to P(Y)$ right and left adjoint to $P(\pi)$. We restrict these maps to the domain $P_{\mathsf{Fix}}(X \times Y)$ to define the right and left adjoints to $P_{\mathsf{Fix}}(\pi)$ as follows. For $\psi \in P_{\mathsf{Fix}}(X \times Y)$,

$$\forall_{\pi}^{\mathsf{Fix}}(\psi) := \mathsf{Fix}_{\Box}(\forall_{\pi}(\psi))$$
$$\exists_{\pi}^{\mathsf{Fix}}(\psi) := \mathsf{Fix}_{\Box}(\exists_{\pi}(\psi)).$$

To show that $\forall_{\pi}^{\mathsf{Fix}}$ is right adjoint to $P_{\mathsf{Fix}}(\pi)$, let $\phi \in P_{\mathsf{Fix}}(Y)$ and suppose $P_{\mathsf{Fix}}(\pi)(\phi) \leq \psi$. Since $P_{\mathsf{Fix}}(\pi)$ is just the restriction $P(\pi)|_{P_{\mathsf{Fix}}(Y)}$, we have $P(\pi)(\phi) \leq \psi$, and since $P(\pi)$ is left adjoint to $\forall_{\pi}$, this means $\phi \leq \forall_{\pi}(\psi)$. But $\psi \in P_{\mathsf{Fix}}(X \times Y)$, so $\forall_{\pi}^{\mathsf{Fix}}(\psi) = \forall_{\pi}(\psi)$ and $\phi \leq \forall_{\pi}^{\mathsf{Fix}}(\psi)$. Since this argument is entirely reversible, the other direction of the bijection holds. A similar argument can be made to show $\exists_{\pi}^{\mathsf{Fix}}$ is left adjoint to $P_{\mathsf{Fix}}(\pi)$.

7.5 Higher-order modal hyperdoctrine

From the hyperdoctrine perspective of "logic over type theory", moving from first-order logic to higher-order logic corresponds to adding more structure to the indexing category $\mathcal{C}$. After specifying the higher-order syntax, we define a higher-order modal hyperdoctrine and prove soundness and completeness.

Higher-order modal logic

We present a higher-order version of a typed modal system S, called HoS. This is achieved by two augmentations to the type structure of TFOL + S. To enable quantification over predicates, we add a special type of propositions to the signature. We also add rules for arrow and finite product types to the term calculus to give a simply typed λ-calculus. These changes follow [Jac99] and [Pit00].

Simply typed $\lambda 1_{\times}$-calculus

In addition to the basic types of our signature Σ we add *compound types* by including the usual type formation rules for arrow (exponent) types $\to$ and finite product types $1, \times$. We also add the usual introduction, elimination and computation rules for terms of these types. For arrow types, these are λ-abstraction, application, and β- and η-conversion. For finite product types, these are pairing, projection, and their conversion rules. Substitution is extended to these terms in the usual way (see [Jac99, Section 2]).

Distinguished type Prop

To be able to quantify over propositions as well as inhabitants of types σ, we add the distinguished type Prop to those listed in the signature. Like the other types, Prop has a list of variables $x, y, z, \ldots$.

On top of the signature, terms-in-context $M : \sigma\ [\Gamma]$ and formulae-in-context $\phi\ [\Gamma]$ are given the same inductive definition as in Section 7.2. Terms of type Prop (in context) are constructed as follows. For each relation symbol $R \subseteq \sigma_1, \ldots, \sigma_n$ in the signature, introduce a corresponding function symbol with codomain Prop, as in

$$R : \sigma_1, \ldots, \sigma_n \to \mathsf{Prop}.$$

Then for $M_1 : \sigma_1\ [\Gamma], \ldots, M_n : \sigma_n\ [\Gamma]$, there is a term $R(M_1, \ldots, M_n)$ of type Prop. Further terms of type Prop are constructed by the logical connectives:

$$\frac{\phi : \mathsf{Prop}\quad [\Gamma] \qquad \psi : \mathsf{Prop}\quad [\Gamma]}{\phi * \psi : \mathsf{Prop}\ [\Gamma]}\ \text{FOR } * \in \{\wedge, \vee, \supset\} \qquad \frac{\phi : \mathsf{Prop}\quad [\Gamma]}{* \phi : \mathsf{Prop}\quad [\Gamma]}\ \text{FOR } * \in \{\neg, \Box\}$$

$$\frac{\phi : \mathsf{Prop}\quad [x : \sigma, \Gamma]}{*_{x:\sigma}\ \phi : \mathsf{Prop}\quad [\Gamma]}\ \text{FOR } * \in \{\forall, \exists\}$$

Substitution over these terms is defined in the usual way (see [Jac99] for full details).

On top of this term calculus, we still have the judgement $\vdash_{\mathsf{HoS}} \phi\ [\Gamma]$, saying that there is a derivation of $\phi\ [\Gamma]$ as governed by the first-order logic rules in Section 7.2. It remains to relate the notion of logical equivalence between formulae[2] to the notion of equality of terms of type Prop via the following rule

$$\text{(PROP)}\ \frac{\vdash_{\mathsf{HoS}} \phi \supset \psi\ [\Gamma] \qquad \vdash_{\mathsf{HoS}} \psi \supset \phi\ [\Gamma]}{\phi = \psi : \mathsf{Prop}\ [\Gamma]}$$

[2]For convenience in the proofs to come, we express it as two separate conditionals rather than the biconditional $\supset\subset$.

in which $\phi = \psi : \mathsf{Prop}$ is judgemental (computational) equality of terms, that is, one term may be converted to the other following the rules of the λ-calculus. Propositions are now terms internal to the type theory.

Modal tripos

7.5.1. DEFINITION. A *modal tripos*, or *higher-order modal hyperdoctrine*, is a modal hyperdoctrine $P : \mathcal{C}^{\mathsf{op}} \to \mathsf{MA}$ where the base category is cartesian closed and there is a *truth-value object* Ω in $\mathcal{C}$ with a natural isomorphism

$$P(C) \simeq \mathrm{Hom}_{\mathcal{C}}(C, \Omega).$$

Modal tripos semantics are given by the following definition.

7.5.2. DEFINITION. Fix a modal tripos $P : \mathcal{C}^{\mathsf{op}} \to \mathsf{MA}$. An interpretation $[\![\text{-}]\!]$ of HoS in P is given by the interpretation in Definition 7.3.3, augmented as follows:

- arrow and finite product types, $\sigma \to \tau$ and $1, \sigma \times \tau$, are interpreted by exponentiation $[\![\tau]\!]^{[\![\sigma]\!]}$ and categorical product $[\![\sigma]\!] \times [\![\tau]\!]$ in $\mathcal{C}$
- the following cases are added to the inductively-defined interpretation of a term: λ-abstraction, λ-application, pairing and projections are interpreted by categorical transpose, evaluation, pairing and projection respectively in $\mathcal{C}$
- the type Prop is assigned to the truth-value object Ω in $\mathcal{C}$, i.e. $[\![\mathsf{Prop}]\!] = \Omega$
- a term $\phi : \mathsf{Prop}\ [\Gamma]$ is assigned to the arrow $[\![\phi]\!] : [\![\Gamma]\!] \to [\![\mathsf{Prop}]\!]$ in $\mathcal{C}$ that corresponds to $[\![\phi\ [\Gamma]]\!] \in P([\![\Gamma]\!])$ via the defining isomorphism of P.

Soundness and completeness

7.5.3. PROPOSITION. *If $\phi\ [\Gamma]$ has a derivation in HoE_X, then it is satisfied in any interpretation in any modal tripos.*

Proof. Fix a modal tripos P and an interpretation $[\![\text{-}]\!]$ in P. With the soundness of modal hyperdoctrine semantics established in Proposition 7.3.5, it remains to show that satisfaction of the *Prop* rule is preserved. Suppose $[\![\phi \supset \psi]\!]$ and $[\![\psi \supset \phi]\!]$ are true in $P([\![\Gamma]\!])$, and so we have

$$\top \leq [\![\phi]\!] \supset [\![\psi]\!] \text{ and } \top \leq [\![\psi]\!] \supset [\![\phi]\!].$$

By (7.2), this is equivalent to $[\![\phi]\!] \leq [\![\psi]\!]$ and $[\![\psi]\!] \leq [\![\phi]\!]$, from which it follows that $[\![\phi]\!] = [\![\psi]\!]$ in $P([\![\Gamma]\!])$. By the isomorphism in the definition of a modal tripos, $[\![\phi]\!], [\![\psi]\!] \in P([\![\Gamma]\!])$ correspond to arrows $[\![\phi]\!], [\![\psi]\!] : [\![\Gamma]\!] \to [\![\mathsf{Prop}]\!]$ in $\mathcal{C}$ that must be equal. These arrows are the interpretations of the terms $\phi : \mathsf{Prop}[\Gamma]$ and $\psi : \mathsf{Prop}[\Gamma]$ respectively, and so we have

$$[\![\phi : \mathsf{Prop}[\Gamma]]\!] = [\![\psi : \mathsf{Prop}[\Gamma]]\!]$$

which is the same as $[\![\phi = \psi : \mathsf{Prop}\ [\Gamma]]\!]$. □

Towards proving completeness, we are interested in the syntactic tripos of HoS, which is defined in the same way as the syntactic hyperdoctrine. Here we prove that it is in fact a modal tripos.

7.5.4. PROPOSITION. *The syntactic hyperdoctrine defined in 7.3.6 is a modal tripos.*

Proof. The existence of finite products and exponentials in $\mathcal{C}$ is guaranteed by the existence of finite product types and function types in the type theory. To show the existence of a truth value object, we need a context up to α-equivalence satisfying the required isomorphism. Take $\Omega = x : \mathsf{Prop}$, noting that this is essentially the same as taking Prop itself when considering $x : \mathsf{Prop}$ as a (single variable) context up to α-equivalence. The required isomorphism then becomes

$$P(\Gamma) \simeq \mathrm{Hom}_{\mathcal{C}}(\Gamma, x : \mathsf{Prop}).$$

By the definition of the syntax, for every formula $\phi\ [\Gamma]$—built from atomic formulae $R(M_1, \ldots, M_n)\ [\Gamma]$ and the logical connectives—there is a corresponding term $\phi : \mathsf{Prop}\ [\Gamma]$—built in the same way from the logical connectives and atomic propositions $R(M_1, \ldots, M_n) : \mathsf{Prop}\ [\Gamma]$. We may consider the term $\phi : \mathsf{Prop}\ [\Gamma]$ as a context morphism in the base category of the modal tripos, that is, as a list of terms of length one, $[\phi] : \Gamma \to \mathsf{Prop}$. This gives the following isomorphism:

$$P(\Gamma) \simeq P(\Gamma, \mathsf{Prop}).$$

To show that this isomorphism is natural, for contexts $\Gamma = \sigma_1, \ldots, \sigma_n$ and $\Gamma' = \tau_1, \ldots, \tau_m$, for any morphism $[t_1, \ldots, t_m] : \Gamma \to \Gamma'$ in the base category, where $t_i : \tau_i\ [\Gamma]$, we require that the following square commutes

$$\begin{tikzcd}
P(\Gamma') \arrow[r, "\mathrm{PaF}'_{\Gamma}"] \arrow[d, "P([t_1,\ldots,t_m])"'] & \mathrm{Hom}_{\mathcal{C}}(\Gamma', \mathsf{Prop}) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}([t_1,\ldots,t_m], x:\mathsf{Prop})"] \\
P(\Gamma) \arrow[r, "\mathrm{PaF}_{\Gamma}"'] & \mathrm{Hom}_{\mathcal{C}}(\Gamma, \mathsf{Prop}),
\end{tikzcd}$$

where PaF (“Propositions as functions”) denotes the isomorphism. This is given by the calculation:

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{C}}([t_1, \ldots, t_m], x : \mathsf{Prop}) \circ \mathrm{PaF}_{\Gamma}(\phi\ [\Gamma']) \\
&= \mathrm{Hom}_{\mathcal{C}}([t_1, \ldots, t_m], x : \mathsf{Prop})(\phi : \mathsf{Prop}\ [\Gamma']) \\
&= \phi[t_1/x_1, \ldots, t_m/x_m] : \mathsf{Prop}\ [\Gamma] \\
&= \mathrm{PaF}_{\Gamma}(\phi[t_1/x_1, \ldots, t_m/x_m][\Gamma]) \\
&= \mathrm{PaF}_{\Gamma} \circ P([t_1, \ldots, t_m])(\phi[\Gamma']).
\end{aligned}$$

□

It is straightforward to see that if $\phi\ [\Gamma]$ is valid in the canonical interpretation in the syntactic tripos, then it is provable in HoS. The standard counter-model argument then immediately gives completeness. Combined with soundness, we obtain the following theorem.

7.5.5. THEOREM. *$\phi\ [\Gamma]$ is provable in HoS iff it is valid in any interpretation in any modal tripos.*

8 Conclusion

Returning to the aims of the thesis, we have successfully extracted crisp type theory as an internal language of a category and used this to relate the category-theoretic and type-theoretic descriptions of the universal uniform fibration from models of homotopy type theory. This required first understanding models of crisp type theory, which we achieved by considering the more general question of how to model the dual-context structure. We also began developing Kripke-Joyal forcing semantics for crisp type theory, as an alternative tool to directly unfolding the interpretation of judgements in the internal crisp type theory $\mathcal{T}_{\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat}$ into the category $\widehat{\mathbb{C}}\downarrow\widehat{\mathbb{C}}_\flat$.

In the additional project on modal hyperdoctrines, we extended these semantics to the weak class of non-normal modal logics and to higher-order modal logics. We also related modal hyperdoctrines to the standard notion of intuitionistic hyperdoctrine via a translation theorem.

We conclude with some future research directions.

8.1 Future work

Interpreting dual-context type theories

The development of the model of dual-context type theory in Chaper 2 was sufficient for proceeding to the motivating problem of the thesis but is limited in scope. For example, we also considered how to formulate the $\flat$ modality as algebraic structure on a fibred natural model and wish to return to this. We are also interested in formalising this as a semantics by specifying a complete syntax of dual-context type theory and proving that it yields an initial fibred natural model.

Kripke-Joyal forcing for crisp type theory

We intend to continue developing the forcing semantics established in Chapter 5 by unfolding the definition of forcing with respect to each of the type-forming operations of the crisp internal language. We did not pursue this here (see Remark 5.2.5) but we are interested in looking for other places where it could be useful.

Universal uniform fibration construction

We demonstrated how to relate the diagrammatic and type-theoretic versions of the universal uniform fibration using our internal language by considering part of the construction. We were limited, however, in which part we could investigate due to the absence of a hierarchy of universes (see Remark 6.3.7). While this was sufficient to illustrate the concept, it could be interesting to include a universe hierarchy and consider other parts of the construction. In another direction, the universal uniform fibration from [LOPS18] is only a universe of *crisp* fibrant types. We wish to consider how our work relates to that of Riley [Ril24], who obtains a classifier of *non-crisp* fibrations.

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