Fibred models of crisp type theory and Kripke-Joyal forcing

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PhD final talk
Foundations cluster seminar
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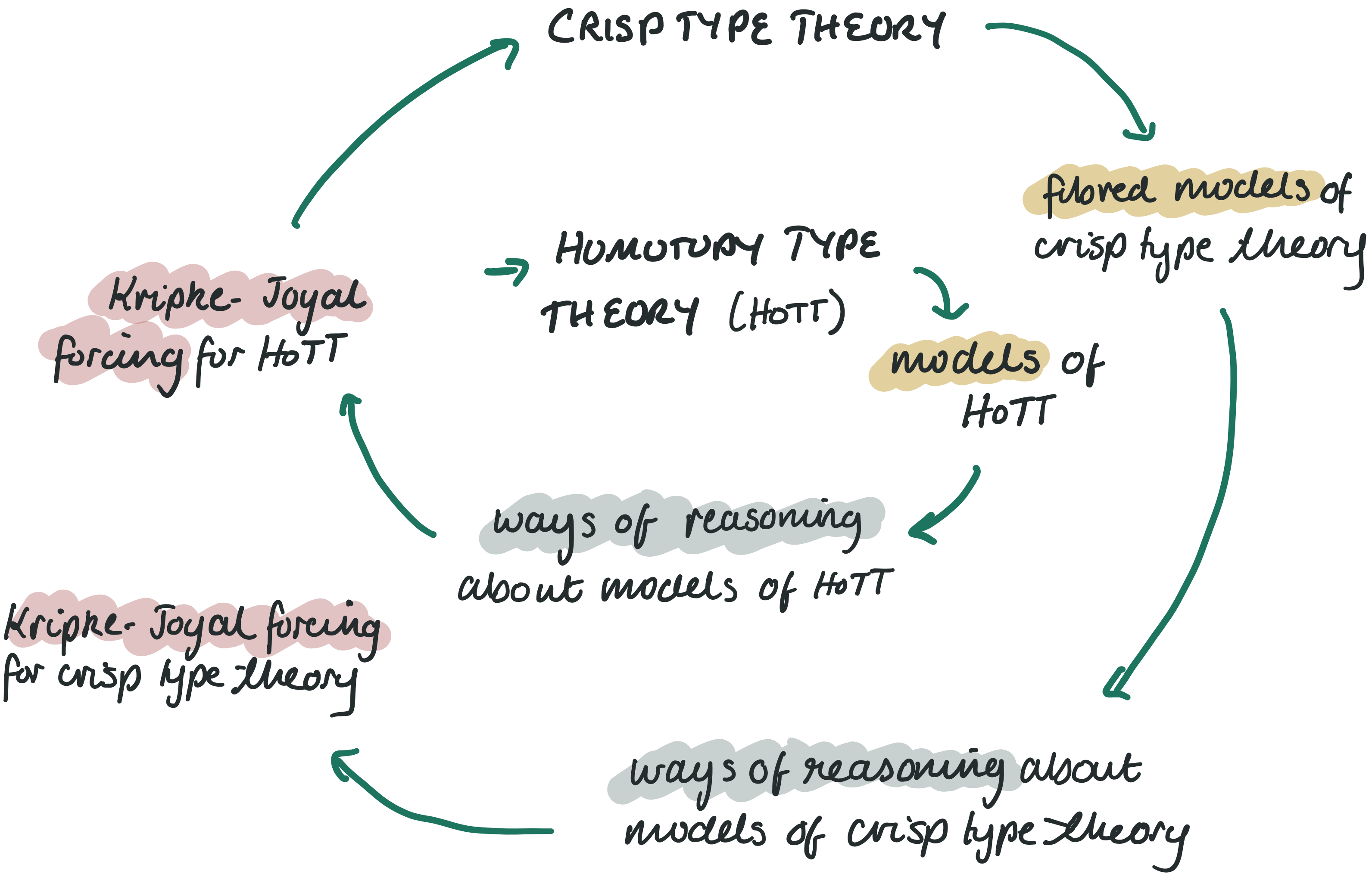
Sketch



CRISP TYPE THEORY

fibred models of crisp type theory

HOMOTOPY TYPE THEORY (HoTT)

models of HoTT

ways of reasoning about models of HoTT

Kripke-Joyal forcing for HoTT

ways of reasoning about models of crisp type theory

Kripke-Joyal forcing for crisp type theory

Identity types

Formation rule

$$\frac{x, y : A}{x =_A y \text{ is a type}}$$

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can be iterated

$$\frac{p, q : x =_A y}{p =_{x =_A y} q \text{ is a type}}$$

Identity types

Formation rule

$$\frac{x, y : A}{x =_A y \text{ is a type}}$$

can be iterated

$$\frac{p, q : x =_A y}{p =_{x =_A y} q \text{ is a type}}$$

and again

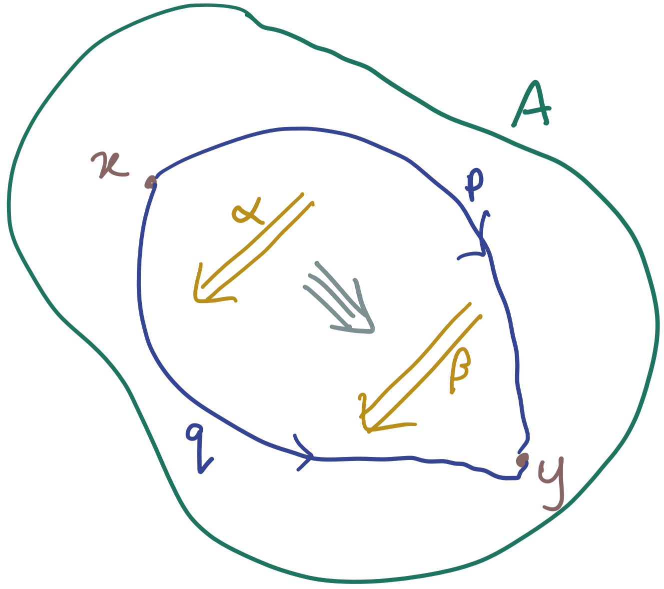
$$\frac{\alpha, \beta : p =_{x =_A y} q}{\alpha =_{p =_{x =_A y} q} \beta \text{ is a type}} \quad \cdots$$

How do we make sense of this?

+ non-uniqueness of identity proofs in the groupoid model
(Hofmann and Streicher 1995)

Intuition

$\alpha =_{p =_{x =_A y} q} \beta$ is a type



A is a space
x is a point
p is a path
α is a homotopy*...

* synthetic space/point/path..

Intuition taken seriously ...

...with a categorical model

(Gambino and Garner 2007, Awodey and Warren 2008)

Rules for identity types

$a : A \vdash \mathrm{refl}(a) : \mathrm{Id}_A(a,a)$

$a : A \vdash c(a) : C(a,a,\mathrm{refl}(a))$

Homotopical interpretation

$$\begin{tikzcd}
A \arrow[rr, tail, "\mathrm{refl}"] \arrow[dr, "s_A"'] & & \mathrm{Id}_A \arrow[dl] \\
& A \times A &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "c"] \arrow[d, tail, "\mathrm{refl}"'] & C \arrow[d] \\
\mathrm{Id}(A) \arrow[r, equal] \arrow[ur, dashed, "J"] & \mathrm{Id}(A)
\end{tikzcd}$$

Defining conditions of a "weak factorisation system"
(wfs)

+ homotopical interpretations for other type constructors
(Voevodsky)

Intuition taken seriously

Consequence the field of Homotopy Type Theory at the intersection of logic / homotopy theory / higher category theory

(the type theory = Martin-Löf type theory
+ univalence axiom
+ homotopy levels
+ higher inductive types)

- comes from studying homotopical models
- we still investigate HoTT by studying models

e.g. Voevodsky 2006 simplical set model

Cohen, Coquand, Huber & Mörtberg 2016 cubical set model

not constructive

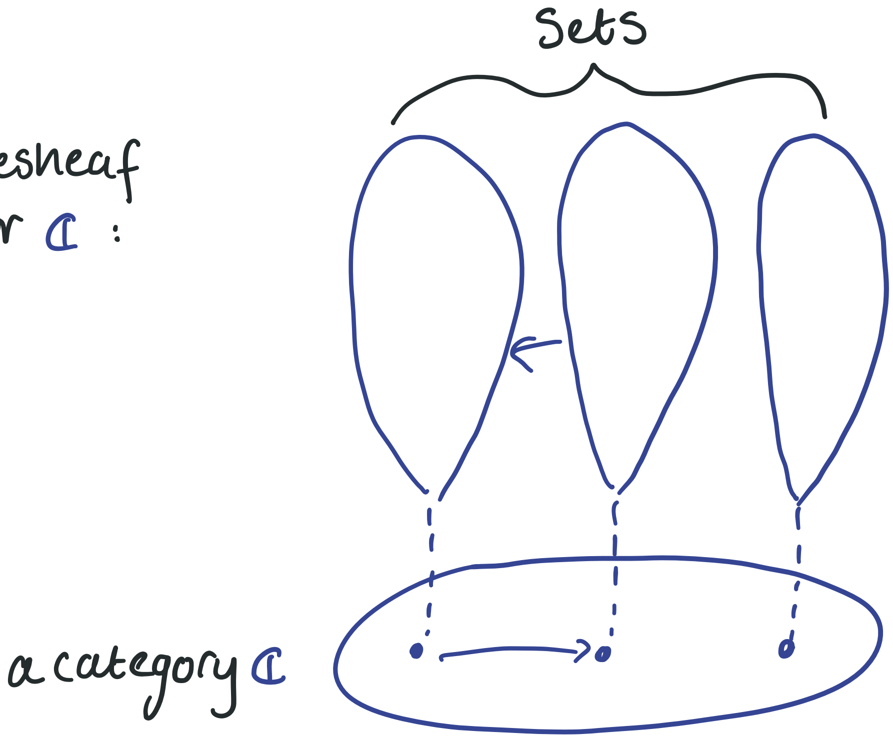
constructive

Models of HoTT

Want: mathematical settings (i.e. categories) with weak factorisation systems

Answer: "presheaf categories"

A presheaf over $\mathbb{C}$:



e.g. let $\mathbb{C}$ be simplices
$\rightsquigarrow$ simplicial sets

let $\mathbb{C}$ be cubes
$\rightsquigarrow$ cubical sets

Working with presheaf-based models

Two ways of working in a presheaf category $\hat{\mathbb{C}}$:

① Category-theoretically via diagrams in $\hat{\mathbb{C}}$

(Awodey, Gambino & Sattler, ...)

② Logically via the "internal type theory" of $\hat{\mathbb{C}}$

(Coquand et al, Orton & Pitts, ...)

Example

a "trivial fibration structure" on ...

(part of a wfs)

① (category-theoretic)

∴ p is a choice of diagonal fillers $j(m,u,v)$

$$\begin{tikzcd} S \arrow[r, "u"] \arrow[d, hook, "m"'] & A \arrow[d, "p"] \\ T \arrow[r, "v"'] \arrow[ur] & X \end{tikzcd}$$

for all $m \in Cof$ such that

$$\begin{tikzcd} t^*(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r, "u"] \arrow[d] & A \arrow[d, "p"] \\ T' \arrow[r, "t"'] \arrow[urr] & T \arrow[r, "v"'] \arrow[ur] & X \end{tikzcd}$$

for all $m \in Cof$, for all $t : T' \longrightarrow T$.

② (type-theoretic)

... $\alpha : X \to U$ is an element

$$t : TFib(\alpha)$$

where

$$TFib(\alpha) = \prod_{\varphi : \Phi} \prod_{v : \alpha^{\{\varphi\}}} \sum_{a : \alpha} v = \lambda(a)$$

Internal type theory

A presheaf category $\hat{\mathbb{C}}$ $\rightsquigarrow$ ingredients of a type theory $\mathcal{T}_{\hat{\mathbb{C}}}$

object Γ $\rightsquigarrow$ context Γ

"small" map

$$\begin{tikzcd} \Gamma.\alpha \arrow[d, "p"'] \\ \Gamma \end{tikzcd}$$

$\rightsquigarrow$ type $\Gamma \vdash \alpha$ type

context extension $\Gamma, x:\alpha \vdash$

section

$$\begin{tikzcd} \Gamma.\alpha \arrow[d, "p"] \\ \Gamma \arrow[u, bend left, "a"] \end{tikzcd}$$

$\rightsquigarrow$ term-in-context

$\Gamma \vdash a : \alpha$

Working with models of HoTT

Example a "trivial fibration structure" on ...

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How do you relate ① and ②?

Interpreting the internal type theory

A presheaf category $\widehat{\mathbb{C}}$ ⟵ ingredients of a type theory $\mathcal{T}_{\widehat{\mathbb{C}}}$

object Γ ⟵ context Γ

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⟵ term-in-context

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Relating diagrammatic and internal reasoning

Method 1: use the standard semantics for type theory in a locally cartesian closed category
(Seely 1984, Hofmann 1994)

⚠ This is tricky for complex types like

$$\mathrm{TFib}(\alpha) = \prod_{\varphi:\Phi} \prod_{v:\alpha^{\{\varphi\}}} \sum_{a:\alpha} v = \lambda(a)$$

Method 2: use the technique of "Kripke-Joyal forcing"
(Awodey, Gambino and Hazratpour 2024)

Kripke-Joyal forcing for type theory

- Awodey, Gambino and Hazratpour 2024

- a technique for testing the validity of a judgement in an internal language in the category

  e.g. the internal type theory of $\hat{\mathbb{C}}$

- "quasi-mechanical", good for iterated type dependencies

- applied to (algebraic) wfs's in constructive models of HoTT

  e.g. trivial fibration structure, but also "universe" of trivial fibrations

The gap

⚠ the "universe of uniform fibrations":

① (category-theoretic)

$$\begin{tikzcd}
Fib^*(id) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Fill(id \circ -)_I \arrow[d] \\
\mathcal{U} \arrow[r, swap, "\eta"] & (\mathcal{U}^I)_I
\end{tikzcd}$$

② (type-theoretic)

impossible!

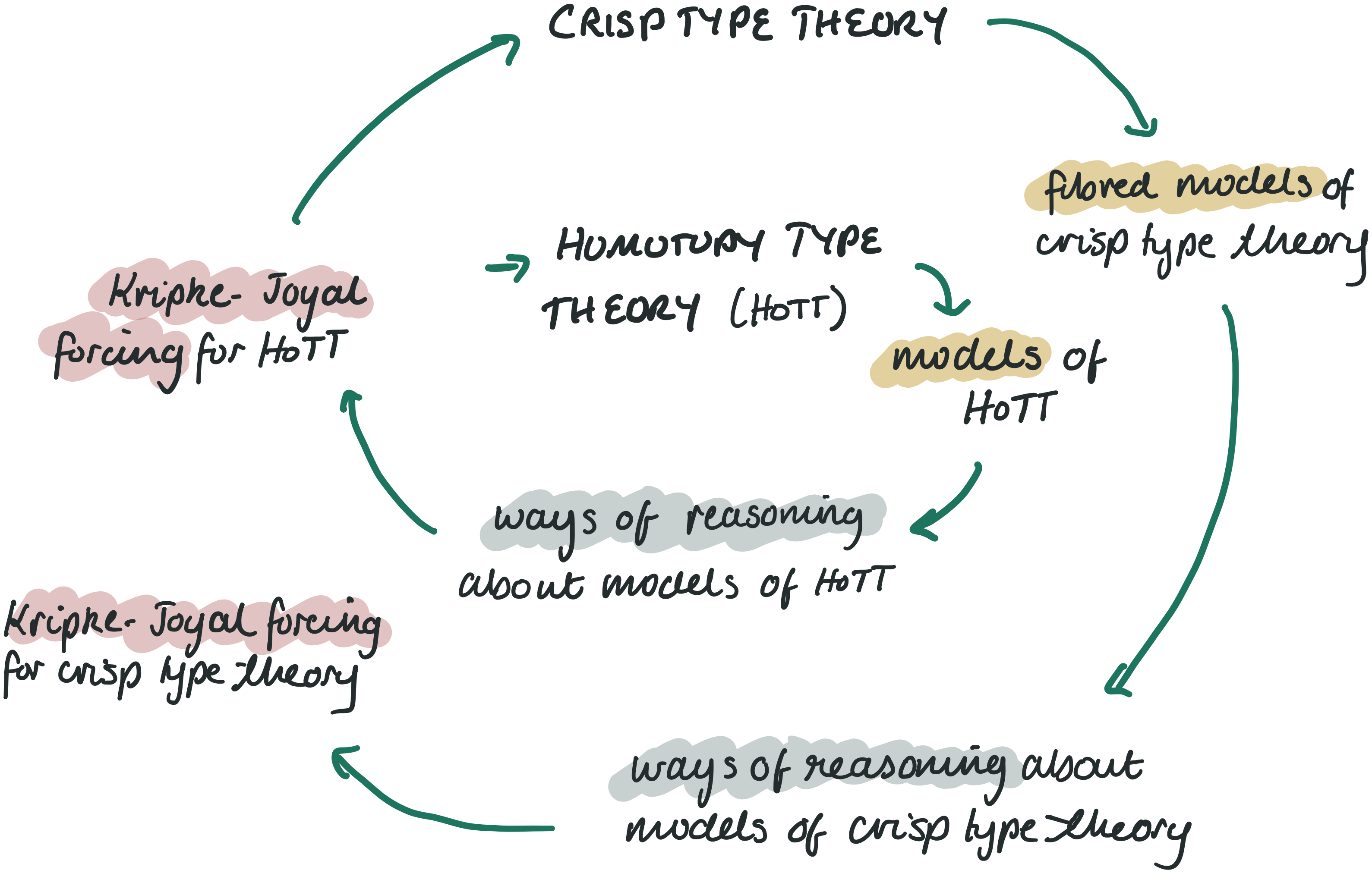
Problem the internal language can't talk about part of the construction

Solution extend the internal language

⇝ "crisp type theory"

(Licata, Orton, Pitts and Spitters (LOPS) 2018)

Recap



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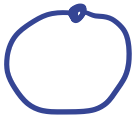
Modalities in HoTT

- crisp type theory is a modal type theory
- a fragment of Shulman's "spatial type theory", part of "real-cohesive HoTT" (2018)
- Used to recover "lost topological information"

e.g. the topological circle $\mathbb{S}^1$ vs. the higher inductive circle S^1

$\{(x,y):\mathbb{R}\times\mathbb{R} \mid x^2+y^2=1\}$



Brouwer's fixed point theorem is trivial for S^1 but not for $\mathbb{S}^1$

Crisp type theory

- Features "flat" modality $\flat A$ $\Box A$
- Features "dual contexts"
  - standard context - $x_1 : \alpha_1, x_2 : \alpha_2, \ldots, x_n : \alpha_n$
  - dual context - $\underbrace{x_1 : \delta_1, \ldots, x_n : \delta_n \mid y_1 : \gamma_1, \ldots, y_m : \gamma_m}_{\Delta \mid \Gamma}$

crisp variables standard variables

$x_1 : \flat\delta_1, \ldots, x_n : \flat\delta_n, y_1 : \gamma_1, \ldots, y_m : \gamma_m$

- crisp types depend only on crisp variables

Crisp type theory

- Two kinds of context extension

① standard context extension

$$\frac{\Delta \mid \Gamma \vdash \alpha \text{ type}}{\Delta \mid \Gamma, x : \alpha \vdash}$$

② extension of the crisp context

$$\frac{\Delta \mid \cdot \vdash \alpha \text{ type}}{\Delta, x :: \alpha \mid \cdot \vdash}$$

+ weakening

$$\frac{\Delta \mid \cdot \vdash \alpha \text{ type} \qquad \Delta \mid \Gamma \vdash \beta \text{ type}}{\Delta, x :: \alpha \mid \Gamma \vdash \beta \text{ type}}$$

Modelling crisp type theory

Intended models (Licata et. al. 2018, from Shulman 2018)

$$\begin{tikzcd}
\text{constant presheaf on } B & \hat{\mathbb{C}} \arrow[loop above, "\flat"] \arrow[d, bend left, "p_*"] \arrow[d, phantom, "\dashv"] & X \arrow[d, mapsto] \\
B \arrow[u, mapsto] & \mathrm{Set} \arrow[u, bend left, "p^*"] & \{1 \xrightarrow{x} X\} \text{ global elements}
\end{tikzcd}$$

$\flat$ is a "comonad" on $\hat{\mathbb{C}}$

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$\flat$ is a "comonad" on $\widehat{\mathbb{C}}$

Data of a comonad

(i) $\Box : \mathcal{C} \to \mathcal{C}$ a functor

(ii) $\varepsilon_C : \Box C \to C$

(iii) $\delta_C : \Box C \to \Box\Box C$

Axioms of modal logic S4

(K) $\Box(A \to B) \to (\Box A \to \Box B)$

(T) $\Box A \to A$

(4) $\Box A \to \Box\Box A$

Internal crisp type theory?

⚠ Licata et al. construct a universal uniform fibration in crisp type theory, but don't present the type theory as an internal language

⇝ can't immediately relate it to the category-theoretic description

Internal crisp type theory

A presheaf category $\widehat{\mathbb{C}}$ idempotent comonad $\flat$		ingredients of crisp type theory
?	$\rightsquigarrow$	dual-context $\Delta \mid \Gamma$
?	$\rightsquigarrow$	type $\Delta \mid \Gamma \vdash \alpha$ type context extension $\Delta \mid \Gamma, x{:}\alpha \vdash$
?	$\rightsquigarrow$	term-in-context $\Delta \mid \Gamma \vdash a : \alpha$

+ two kinds of context extension...

Internal crisp type theory

Our approach

1) zoom out - how can we model the features of a dual-context type theory?

   e.g. • context dependence of $\Delta \mid \Gamma$
   - two kinds of context extension

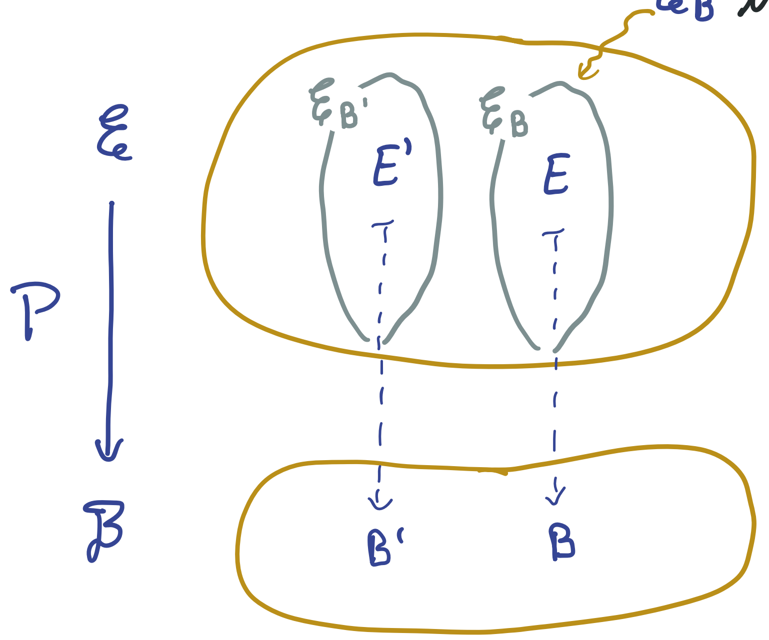
2) zoom back in - how does $\hat{\mathbb{C}}$, $\flat$ admit such a model?

3) use this understanding to extract an internal crisp type theory

Modelling dual-context type theory

For a context $\Delta|\Gamma$, want to capture the dependency of Γ on Δ.

$\mathcal{E}_B$ is the "fibre category" over B



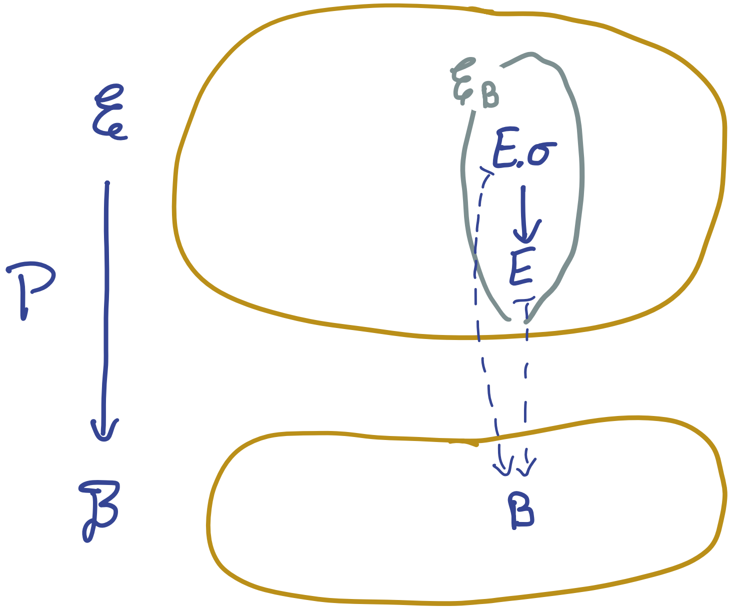
$E \in \mathcal{E}_B \leftrightsquigarrow \Delta|\Gamma$

$B \in \mathcal{B} \leftrightsquigarrow \Delta$

Modelling dual-context type theory

Regular context extension:

$$\frac{\Delta \mid \Gamma \vdash \sigma \ \mathrm{type}}{\Delta \mid \Gamma, \sigma \vdash}$$



$E.\sigma \leftrightsquigarrow \Delta \mid \Gamma, \sigma \vdash$

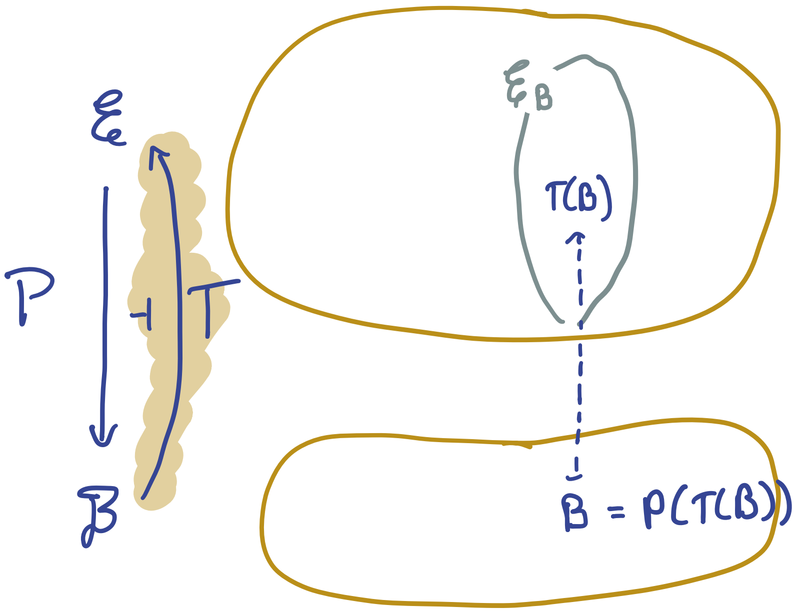
$E \leftrightsquigarrow \Delta \mid \Gamma \vdash$

$B \leftrightsquigarrow \Delta \vdash$

Modelling dual-context type theory

Empty second context zone:

$$\begin{array}{c} \Delta \mid \bullet \vdash \\ \hline\hline \Delta \vdash_{\mathcal{B}} \end{array}$$



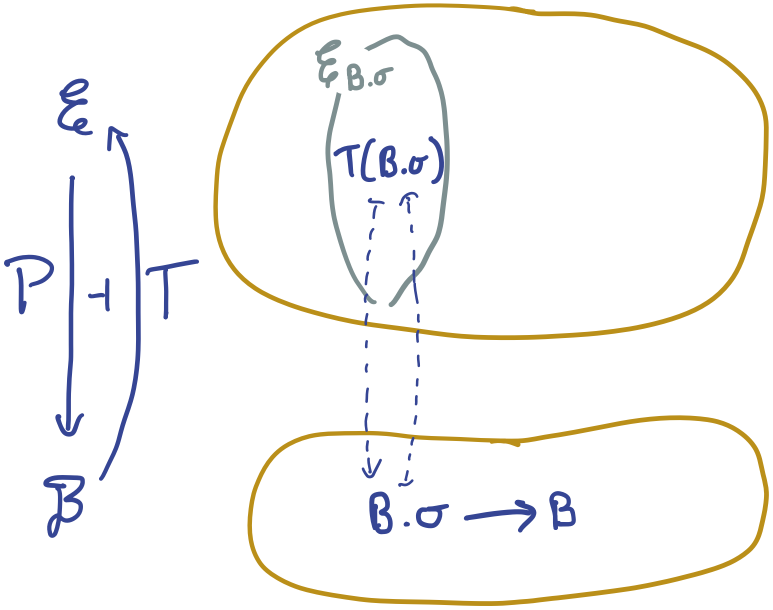
$T(B) \leftrightsquigarrow \Delta \mid \bullet \vdash$

$B \leftrightsquigarrow \Delta \vdash$

Modelling dual-context type theory

Extension of the first context zone:

$$\frac{\Delta \mid \cdot \vdash \sigma \ \mathsf{type}}{\Delta, \sigma \mid \cdot \vdash}$$



$T(B.\sigma) \leftrightsquigarrow \Delta, \sigma \mid \cdot \vdash$

$B \leftrightsquigarrow \Delta \vdash$

$B.\sigma \leftrightsquigarrow \Delta, \sigma \vdash$

Fibred natural model of dual-context type theory

Idea Equip

(i) the base category, and
(ii) each fibre

with the structure to model a type theory.

e.g. Awodey's "natural models" (2016)

These structures should be related, i.e.

$$\Delta \vdash_{\mathcal{B}} \sigma \text{ type} \quad \text{“}=\text{”} \quad \Delta \mid \bullet \vdash_{\mathcal{E}} \sigma \text{ type}$$

⇝ "Fibred natural models of dual-context type theory" given by a functor $p : \mathcal{E} \to \mathcal{B}$ + axioms.

Zooming back in

Recall the intended models are categories with idempotent comonads (e.g. $\widehat{\mathbb{C}}$, $\flat$)

Theorem $\widehat{\mathbb{C}}$, $\flat$ gives rise to a fibred natural model.

$$\begin{tikzcd}
\mathcal{E} := \widehat{\mathbb{C}} \downarrow \widehat{\mathbb{C}}_\flat \arrow[d, "\mathrm{cod}"] & \\
\mathcal{B} := \widehat{\mathbb{C}}_\flat & \text{full subcategory of } X \in \widehat{\mathbb{C}} \text{ with } \flat X = X \arrow[l, squiggly]
\end{tikzcd}$$

Internal crisp type theory

the comma category $\hat{\mathbb{C}} \downarrow \hat{\mathbb{C}}_\flat$ ⇝ ingredients of crisp type theory

objects
$$\begin{tikzcd}
\Theta \arrow[d, "\Delta|\Gamma"] \\
\Delta
\end{tikzcd}$$
⇝ contexts $\Delta|\Gamma$

"vertical" small maps
$$\begin{tikzcd}
& \Theta.\alpha \arrow[dl, "p"'] \arrow[d] \\
\Theta \arrow[d] & \Delta \arrow[dl, equal] \\
\Delta &
\end{tikzcd}$$
$p \in \mathcal{S}$

⇝ types $\Delta|\Gamma \vdash \alpha \text{ type}$

context extension $\Delta|\Gamma, x{:}\alpha \vdash$

"horizontal" small maps
$$\begin{tikzcd}
\Theta * \alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta \arrow[d] \\
\Delta.\alpha \arrow[r] & \Delta
\end{tikzcd}$$
⇝ crisp context extension

$$\frac{\Delta|\cdot \vdash \alpha \text{ type} \qquad \Delta|\Gamma \vdash}{\Delta, x{::}\alpha \,|\, \Gamma \vdash}$$

Internal crisp type theory

- The internal type theory of $\hat{\mathbb{C}}\downarrow\hat{\mathbb{C}}_b$, called $\mathcal{T}_{\hat{\mathbb{C}}\downarrow\hat{\mathbb{C}}_b}$, supports
  - standard Π and Σ types
  - crisp Π-types

    ← used for universal uniform fibration

- In $\mathcal{T}_{\hat{\mathbb{C}}}$, Π-types come from right adjoint to pullback along small maps

- In $\mathcal{T}_{\hat{\mathbb{C}}\downarrow\hat{\mathbb{C}}_b}$, right adjoint to pullback along –
  - vertical small maps $\leftrightsquigarrow$ standard Π-types
  - horizontal small maps $\leftrightsquigarrow$ crisp Π-types

Theorem Crisp type theory is a subtheory of $\mathcal{T}_{\hat{\mathbb{C}}\downarrow\hat{\mathbb{C}}_b}$

Application to models of HoTT

Returning to the universe of uniform fibrations

We can relate the different descriptions:

$$\begin{tikzcd} \text{description in } \hat{\mathcal{C}} \arrow[r, leftrightarrow] & \text{description in } \hat{\mathcal{C}}\downarrow\hat{\mathcal{C}}_b \arrow[r, leftrightarrow] & \text{description in } \hat{\mathcal{T}}_{\hat{\mathcal{C}}\downarrow\hat{\mathcal{C}}_b} \arrow[r, leftrightarrow] & \text{description in crisp type theory} \end{tikzcd}$$

(Awodey et al. 2021)

(Licata et al. 2018)

$\underbrace{\hspace{6em}}$ uses crisp Π-types

Overview of contributions

1) developed fibred model of dual-context type theory
2) specialised to models of crisp type theory
3) extracted crisp type theory as the internal language of a category
4) developed Kripke-Joyal forcing for crisp type theory
5) related (parts of) the category-theoretic and type-theoretic descriptions of the universe of uniform fibrations

Future work

- finish formulating the $\flat$-modality as algebraic structure on a fibred natural model
- formalise the model as semantics
  i.e. specify syntax and prove that it yields an initial such model
- relate the rest of the category-theoretic and type-theoretic descriptions of the universe of uniform fibrations
  - was limited by not setting up a hierarchy of universes in the internal language.
- look for applications of the Kripke-Joyal forcing for crisp type theory

Thank you