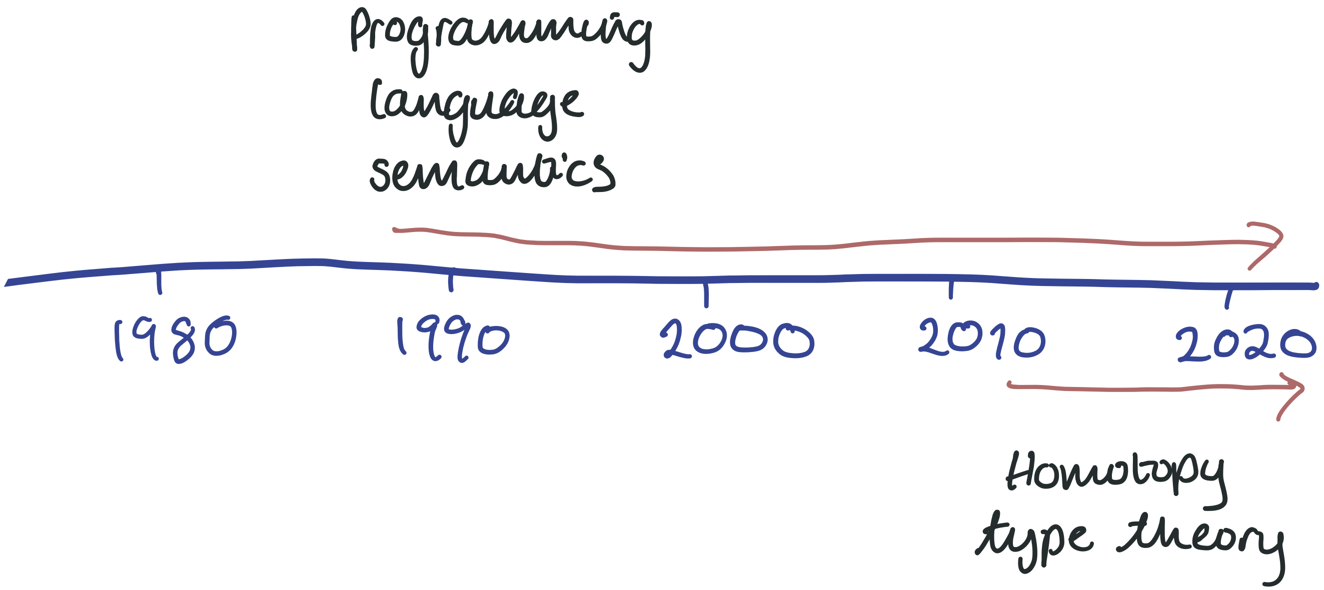
Who cares about modal type theory?

Florrie Verity

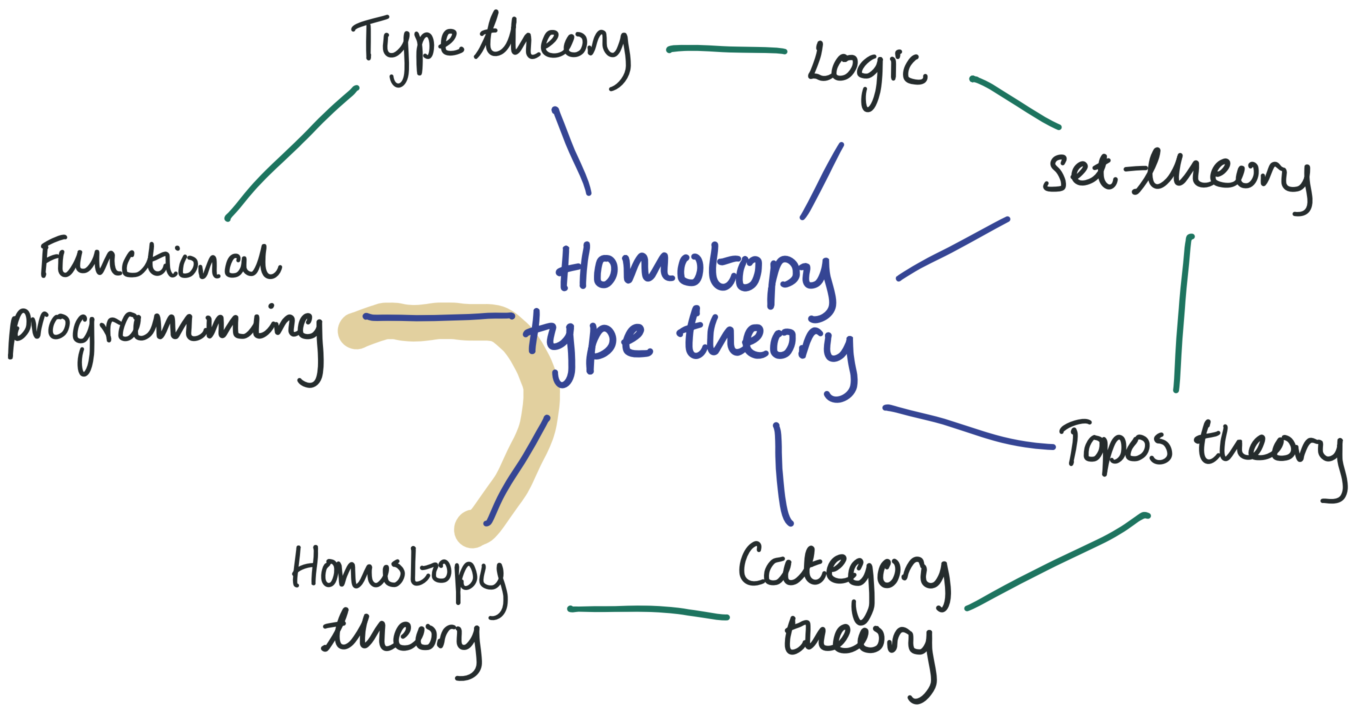
Computing foundations seminar, ANU
13 November 2023

Modal type theory by field



'HoTT Subject map'

- Paige Randall North



Plan

1) Homotopy type theory and modalities

2) Crisp type theory
   - the type theory
   - its semantics

Part one

Identity types

The formation rule $\dfrac{x, y : A}{\mathrm{Id}_A(x,y) \text{ is a type}}$ can be iterated –

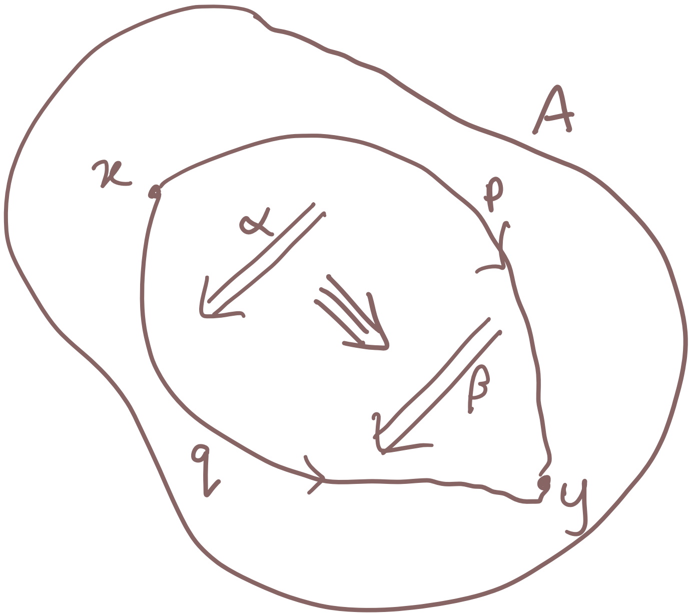
- $$\dfrac{p, q : \mathrm{Id}_A(x,y)}{\mathrm{Id}_{\mathrm{Id}_A(x,y)}(p,q) \text{ is a type}}$$
- $$\dfrac{\alpha, \beta : \mathrm{Id}_{\mathrm{Id}_A(x,y)}(p,q)}{\mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A(x,y)}(p,q)}(\alpha,\beta) \text{ is a type}}$$ and so on.

How do we make sense of this?

– Hofmann and Streicher 1995, $p, q : \mathrm{Id}_A(x,y) \not\Rightarrow p = q$

Intuition

$$\frac{\alpha, \beta : \mathrm{Id}_{\mathrm{Id}_A(x,y)}(p,q)}{\mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A(x,y)}(p,q)}(\alpha,\beta) \text{ is a type}}$$



A is a space
x is a point
p is a path
α is a homotopy*

* synthetic space/point/path..

Intuition taken seriously

Rules for identity types

$$a : A \vdash \mathrm{refl}(a) : \mathrm{Id}_A(a,a)$$

$$a : A \vdash c(a) : C(a,a,\mathrm{refl}(a))$$

Homotopical interpretation

$$\begin{tikzcd} A \arrow[rr, tail, "\mathrm{refl}"] \arrow[dr, "s_A"'] & & \mathrm{Id}_A \arrow[dl] \\ & A \times A & \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, tail, "\mathrm{refl}"'] & C \arrow[d, two heads] \\ \mathrm{Id}(A) \arrow[r, equal] \arrow[ur, dashed, "j"] & \mathrm{Id}(A) \end{tikzcd}$$

Defining conditions of a weak factorisation system in homotopical algebra

Consequence

development of HoTT

(= Martin Löf type theory
+ "univalence axiom"
+ a focus on "homotopy levels"
+ "higher inductive types")

Models of HoTT

- HoTT has models in "presheaf categories", a setting with an abundance of homotopical model structures
  - simplicial sets (Voevodsky)
  - cubical sets (Coquand, Orton & Pitts, Awodey)

    constructive!

- Two ways of working in a presheaf category $\hat{\mathbb{C}}$:

  ① Category-theoretically via diagrams in $\hat{\mathbb{C}}$
  (Awodey, Gambino & Sattler, ...)

  objects and structure-preserving maps

  ② Logically via the "internal type theory" of $\hat{\mathbb{C}}$
  (Coquand et al, Orton & Pitts, ...)

Internal logic

universe of terms

universe of types

Remark $\widehat{\mathbb{C}}$ has two special objects $\widetilde{\mathcal{U}}$ and $\mathcal{U}$, and a map between them, $ty : \widetilde{\mathcal{U}} \to \mathcal{U}$.

A presheaf category $\widehat{\mathbb{C}}$ — ingredients of a type theory $\mathcal{T}_{\widehat{\mathbb{C}}}$

object Γ $\rightsquigarrow$ context Γ

map $\Gamma \xrightarrow{\alpha} \mathcal{U}$ $\rightsquigarrow$ type-in-context $\Gamma \vdash \alpha$ is a type

diagram

$$\begin{tikzcd}
 & \widetilde{\mathcal{U}} \arrow[d, "ty"] \\
\Gamma \arrow[ur, "a"] \arrow[r, "\alpha"'] & \mathcal{U}
\end{tikzcd}$$

$\rightsquigarrow$ term-in-context $\Gamma \vdash a : \alpha$

"pullback" diagram

$$\begin{tikzcd}
\Gamma.\alpha \arrow[r, "q_\alpha"] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "ty"] \\
\Gamma \arrow[r, "\alpha"'] & \mathcal{U}
\end{tikzcd}$$

$\rightsquigarrow$ context extension $\Gamma, \alpha \vdash q_\alpha : \alpha[p_\alpha]$

Working with models of HoTT

Example a "trivial fibration structure" on ...

① (category-theoretic)

... p is a choice of diagonal fillers $j(m,u,v)$

$$\begin{tikzcd}
S \arrow[r, "u"] \arrow[d, tail, "m"'] & A \arrow[d, "p"] \\
T \arrow[r, "v"'] \arrow[ur] & X
\end{tikzcd}$$

for all $m \in Cof$ such that

$$\begin{tikzcd}
t^*(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r, "u"] \arrow[d] & A \arrow[d, "p"] \\
T' \arrow[r, "t"'] \arrow[urr, bend left=10] & T \arrow[r, "v"'] \arrow[ur] & X
\end{tikzcd}$$

for all $m \in Cof$, for all $t : T' \to T$.

② (type-theoretic)

... $\alpha : X \to U$ is an element

$$t : TFib(\alpha)$$

where

$$TFib(\alpha) = \prod_{\varphi : \Phi} \prod_{v : \alpha^{\{\varphi\}}} \sum_{a : \alpha} v = \lambda(a)$$

? How do you relate ① and ②?

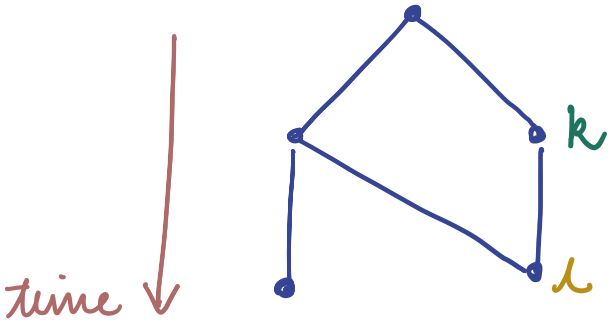
Answer: a generalisation of

Beth-Kripke semantics for intuitionistic logic

Sets of valid formulae at a stage

$F(k) \subseteq F(l)$

Poset of stages



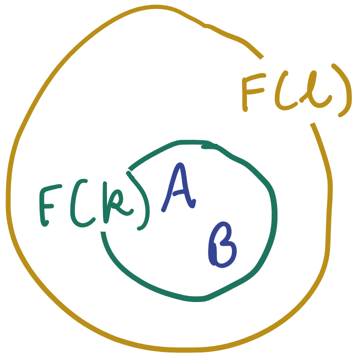
$k \leq l$

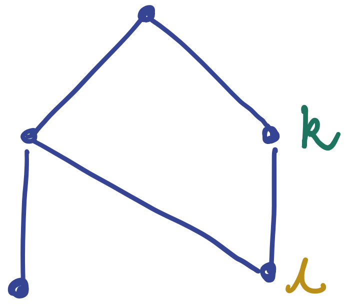
Beth–Kripke semantics

"Forcing" relation $k \Vdash A$

plus conditions for compound formulae, e.g.

$k \Vdash A \wedge B$ iff $k \Vdash A$ and $k \Vdash B$





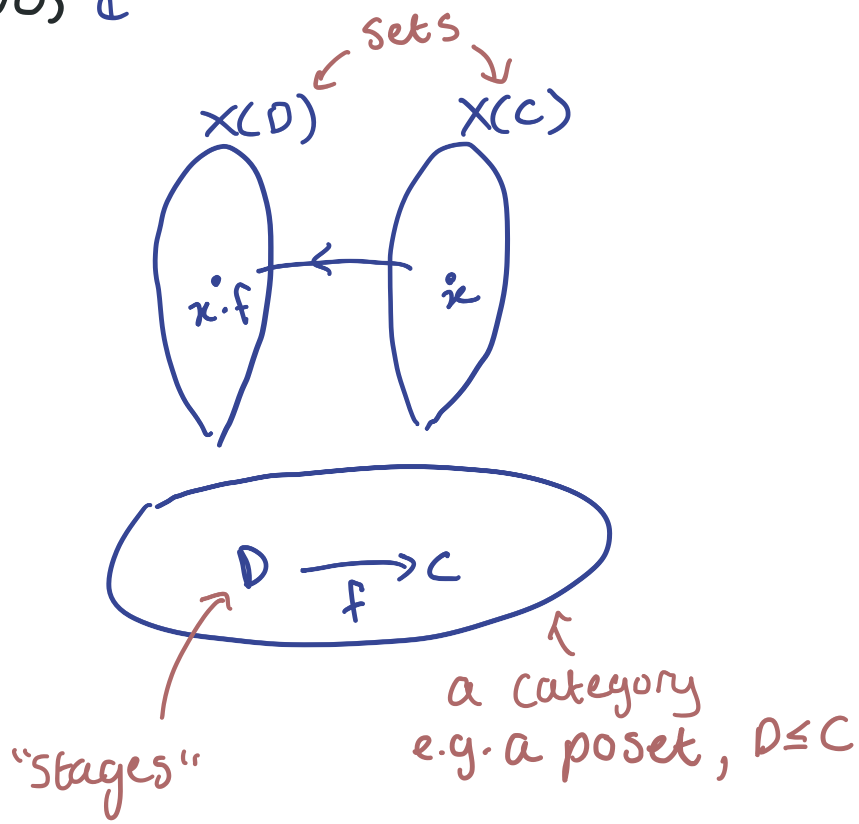
Kripke-Joyal semantics

- a generalisation for higher-order logic

- setting: a topos ∘ ○ ○ all the abstract structure of sets, but "richer"

e.g. a category of presheaves $\hat{\mathbb{C}}$

a presheaf X in $\hat{\mathbb{C}}$
is a "variable set"



Kripke-Joyal semantics for HoTT

-Awodey, Gambino and Hazratpour, 2021

- an extension of KJ semantics to the internal type theory of a presheaf category

Forcing definition

$c \Vdash a_x : \alpha(x)$

means this diagram commutes :

$$\begin{tikzcd} & & \tilde{\mathcal{U}} \arrow[d, "ty"] \\ \text{よ}c \arrow[r, "x"'] \arrow[urr, dashed, "a_x"] & X \arrow[r, "\alpha"'] & \mathcal{U} \end{tikzcd}$$

+ conditions for type formers,

e.g. $c \Vdash t : (\alpha \times \beta)(x)$ iff $c \Vdash a : \alpha(x)$ and $c \Vdash b : \alpha(x)$

Example

a "trivial fibration structure" on . . .

① (category-theoretic)

∴ p is a choice of diagonal fillers $j(m,u,v)$

$$\begin{tikzcd}
S \arrow[r, "u"] \arrow[d, tail, "m"'] & A \arrow[d, "p"] \\
T \arrow[r, "v"'] \arrow[ur] & X
\end{tikzcd}$$

for all $m \in Cof$ such that

$$\begin{tikzcd}
t^*(S) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & S \arrow[r, "u"] \arrow[d] & A \arrow[d, "p"] \\
T' \arrow[r, "t"'] \arrow[urr] & T \arrow[r, "v"'] \arrow[ur] & X
\end{tikzcd}$$

for all $m \in Cof$, for all $t : T' \to T$.

② (type-theoretic)

$\ldots \alpha : X \to \mathcal{U}$ is an element

$t : TFib(\alpha)$

where

$$TFib(\alpha) = \prod_{\varphi : \Phi} \prod_{v : \alpha^{\{\varphi\}}} \sum_{a : \alpha} v = \lambda(a)$$

The gap

⚠ the "universe of uniform fibrations":

① (category-theoretic)

$$\begin{tikzcd}
\mathrm{Fib}^*(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fill}(\alpha \circ x)_I \arrow[d] \\
X \arrow[r, "\eta"'] & (X^I)_I
\end{tikzcd}$$

② (type-theoretic)

impossible!

Solution

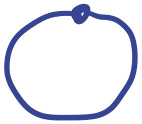
- extend type theory with the modal operator of "crisp type theory" (Licata, Orton, Pitts & Spitters, 2018)
- formulate Kripke-Joyal semantics for this type theory (present work)

Modalities elsewhere in HoTT

Used to recover "lost topological information"

e.g. the topological circle $\mathbb{S}^1$ vs. the higher inductive circle S^1

$\{(x,y) : \mathbb{R}\times\mathbb{R} \mid x^2+y^2=1\}$



- Brouwer's fixed point theorem is trivial for S^1 but not for $\mathbb{S}^1$

Part two

"A judgemental reconstruction of modal logic"

– Pfenning and Davies, 2001

Basic judgements in logic

1) A is a proposition

we know *what counts* as a verification of A

2) A is true

we know *how* to verify A

(presupposes A is a proposition)

used in inference rules to explain connectives e.g. conjunction

$$\frac{A \text{ prop} \quad B \text{ prop}}{A \wedge B \text{ prop}} \quad \text{Formation}$$

$$\frac{A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}} \quad \text{Introduction}$$

$$\left.\begin{array}{c} \dfrac{A \wedge B \text{ true}}{A \text{ true}} \\[2ex] \dfrac{A \wedge B \text{ true}}{B \text{ true}} \end{array}\right\} \text{Elimination}$$

Hypothetical judgements

- to explain the connective $\Rightarrow$, we need another form of judgement, written:

$$\underbrace{J_1, \ldots, J_n}_{\text{“hypotheses”}} \vdash J$$

J assuming J_1 through J_n

e.g. $A_1 \text{ true}, \ldots, A_n \text{ true} \vdash A \text{ true}$

- this allows us to introduce implications with the rule:

$$\frac{\Gamma, A \text{ true} \vdash B \text{ true}}{\Gamma \vdash A \Rightarrow B \text{ true}}$$

we know how to verify $A \Rightarrow B$ if we know how to verify B under hypothesis “A true”

- we may as well write our other rules in this judgement form, e.g.

$$\frac{A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}} \quad \rightsquigarrow \quad \frac{\Gamma \vdash A \text{ true} \quad \Gamma \vdash B \text{ true}}{\Gamma \vdash A \wedge B \text{ true}}$$

Rethinking our judgements...

Recall the second basic judgement

2) A is true

we know how to verify A

Let's give names to verifications and replace the above judgement with

$$M : A$$

"M is a proof of proposition A"

"M is a term of type A

For hypothetical judgements, we name our hypothesised proof/term with a variable:

$$x : A$$

... leads to type theory

Example Conjunction

- Formation rule
$$\frac{A\ \text{prop} \qquad B\ \text{prop}}{A \wedge B\ \text{prop}}$$

- Introduction rule
$$\frac{A\ \text{true} \quad B\ \text{true}}{A \wedge B\ \text{true}} \quad \rightsquigarrow \quad \frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \wedge B}$$

- Elimination rule
$$\frac{A \wedge B\ \text{true}}{A\ \text{true}} \quad \rightsquigarrow \quad \frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash \text{fst}\ M : A}$$
$$\frac{A \wedge B\ \text{true}}{B\ \text{true}} \quad \rightsquigarrow \quad \frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash \text{snd}\ M : B}$$

- Computation rules
$$\text{fst}\langle M, N \rangle \Rightarrow_R M$$
$$\text{snd}\langle M, N \rangle \Rightarrow_R N$$
$$M : A \wedge B \Rightarrow_E \langle \text{fst}\ M, \text{snd}\ M \rangle$$

relate intro and elim rules

Pfenning and Davies' idea.

use this methodology of analysing judgements to incorporate modality in a type theory

Step 1) Introduce a third basic judgement

Definition (Validity)

1) If $\bullet \vdash A \text{ true}$ then $A \text{ valid}$.
2) if $A \text{ valid}$ then $\Gamma \vdash A \text{ true}$.

necessarily true

This may be used in hypothetical judgements

$$B_1 \text{ valid}, \dots, B_m \text{ valid} \mid A_1 \text{ true}, \dots, A_n \text{ true} \vdash A \text{ true},$$

abbreviated

$$\Delta \mid \Gamma \vdash A \text{ true}.$$

Step 2) Internalise this judgement as a proposition

- Formation rule $$\frac{A \text{ prop}}{\Box A \text{ prop}}$$

- Introduction rule $$\frac{\Delta \mid \bullet \vdash A \text{ true}}{\Delta \mid \Gamma \vdash \Box A \text{ true}}$$

(follows from the definition of validity, updated with split contexts.

1) If $\Delta \mid \bullet \vdash A$ true then A valid.
2) if A valid then $\Delta \mid \Gamma \vdash A$ true.)

- Elimination rule $$\frac{\Delta \mid \Gamma \vdash \Box A \text{ true} \qquad \Delta, A \text{ valid} \mid \Gamma \vdash C \text{ true}}{\Delta \mid \Gamma \vdash C \text{ true}}$$

Step 3) Perform the same move as before to "term : type" judgements

- Formation rule

$$\frac{A\ \text{prop}}{\Box A\ \text{prop}}$$

- Introduction rule

$$\frac{\Delta \mid \cdot \vdash A\ \text{true}}{\Delta \mid \Gamma \vdash \Box A\ \text{true}} \quad \rightsquigarrow \quad \frac{\Delta \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash \text{box}\ M : \Box A}$$

- Elimination rule

$$\frac{\Delta \mid \Gamma \vdash \Box A\ \text{true} \qquad \Delta, A\ \text{valid} \mid \Gamma \vdash C\ \text{true}}{\Delta \mid \Gamma \vdash C\ \text{true}}$$

$$\rightsquigarrow \quad \frac{\Delta \mid \Gamma \vdash M : \Box A \qquad \Delta, \overset{\text{modal variable}}{\underline{u :: A}} \mid \Gamma \vdash N : C}{\Delta \mid \Gamma \vdash \text{let box}\ u = M\ \text{in}\ N : C}$$

- Computation rules

$$\text{let box}\ u = \text{box}\ M\ \text{in}\ N \Rightarrow_R N[\![M/u]\!] \quad \text{replace all instances of } u \text{ in } N \text{ with } M$$

$$M : \Box A \Rightarrow_E \text{let box}\ u = M\ \text{in}\ (\text{box}\ u)$$

Moving to Crisp type theory

- Crisp type theory is dependently-typed
- Terminology changes

box modality $\Box A$ $\rightsquigarrow$ flat modality $\flat A$

validity hypotheses $u :: A$ $\rightsquigarrow$ "crisp" hypotheses

"crisp context / context of crisp variables" $\rightsquigarrow$ $\Delta \mid \Gamma$ $\leftsquigarrow$ "non-crisp context, context of non-crisp variables"

Context extension

- Two kinds of context extension

① standard context extension

$$\frac{\Delta \mid \Gamma \vdash \alpha \ \mathsf{type}}{\Delta \mid \Gamma, x : \alpha \vdash}$$

② extension of the crisp context

$$\frac{\Delta \mid \bullet \vdash \alpha \ \mathsf{type}}{\Delta, x : \alpha \mid \bullet \vdash}$$

Modelling dependent type theory

Recall the "internal logic" of $\widehat{\mathbb{C}}$

A presheaf category $\widehat{\mathbb{C}}$ $\rightsquigarrow$ **ingredients of a type theory $\mathcal{T}_{\widehat{\mathbb{C}}}$**

object Γ $\rightsquigarrow$ context Γ

map $\Gamma \xrightarrow{\alpha} \mathcal{U}$ $\rightsquigarrow$ type-in-context

$\Gamma \vdash \alpha$ is a type

diagram

$$\begin{tikzcd}
 & \widetilde{\mathcal{U}} \arrow[d, "ty"] \\
\Gamma \arrow[ur, "a"] \arrow[r, "\alpha"'] & \mathcal{U}
\end{tikzcd}$$

$\rightsquigarrow$ term-in-context

$\Gamma \vdash a : \alpha$

"pullback" diagram

$$\begin{tikzcd}
\Gamma.\alpha \arrow[r, "q_\alpha"] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "ty"] \\
\Gamma \arrow[r, "\alpha"'] & \mathcal{U}
\end{tikzcd}$$

$\rightsquigarrow$ context extension

$\Gamma, \alpha \vdash q_\alpha : \alpha[p_\alpha]$

Modelling dependent type theory

We can go the other way -

Category $\mathcal{D}$ with appropriate structure ⟵ **Ingredients of a type theory $\mathcal{T}$**

object Γ ⟵ context Γ

map $\Gamma \xrightarrow{\alpha} \mathcal{U}$ ⟵ type-in-context $\Gamma \vdash \alpha$ is a type

diagram

$$\begin{tikzcd} & \tilde{\mathcal{U}} \arrow[d, "ty"] \\ \Gamma \arrow[ur, "a"] \arrow[r, "\alpha"'] & \mathcal{U} \end{tikzcd}$$

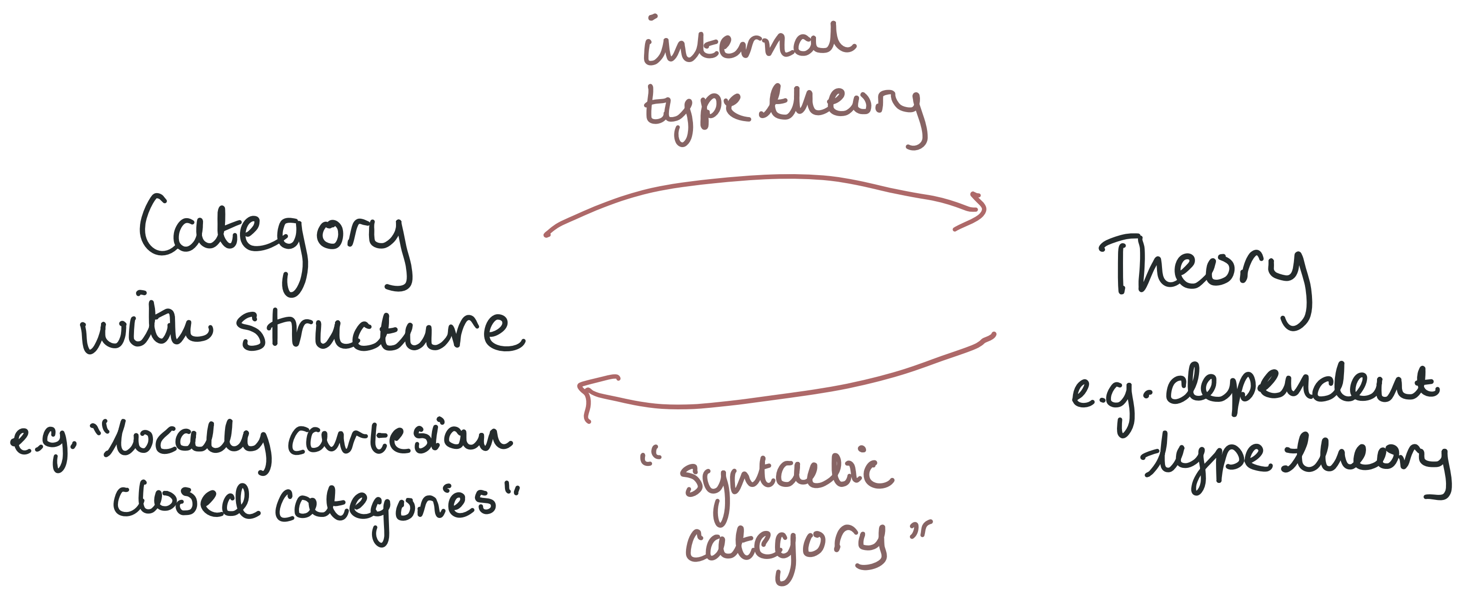
⟵ term-in-context $\Gamma \vdash a : \alpha$

"pullback" diagram

$$\begin{tikzcd} \Gamma.\alpha \arrow[r, "q_\alpha"] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d, "ty"] \\ \Gamma \arrow[r, "\alpha"'] & \mathcal{U} \end{tikzcd}$$

⟵ context extension $\Gamma, \alpha \vdash q_\alpha : \alpha[p_\alpha]$

"Functorial semantics"



internal type theory

Category with structure

e.g. "locally cartesian closed categories"

Theory

e.g. dependent type theory

" syntactic category "

we have a "correspondence"

e.g. object Γ in $\mathcal{D}$ $\leftrightsquigarrow$ context Γ in $\mathcal{T}$

N.B. subtleties arise from type theories being "stricter" than categories

Modelling crisp type theory

- Conjectured model in Licata et. al. (2018), from Shulman (2018)

$$\begin{tikzcd}
\text{constant presheaf on } B & \hat{\mathbb{C}} \arrow[loop above, "\flat"] \arrow[d, bend left, "p_*"] & X \arrow[d, mapsto] \\
B \arrow[u, mapsto] & \mathrm{Set} \arrow[u, bend left, "p^*"] \arrow[u, phantom, "\dashv"] & \{1 \overset{x}{\Rightarrow} X\} \text{ global elements}
\end{tikzcd}$$

$\flat$ is a "comonad" on $\hat{\mathbb{C}}$

Data of a comonad

(i) $\Box : \mathcal{C} \to \mathcal{C}$ a functor

(ii) $\varepsilon : \Box \Rightarrow id_{\mathcal{C}}$

(iii) $\delta : \Box \Rightarrow \Box\Box$

Axioms of modal logic S4

(K) $\Box(A \to B) \to (\Box A \to \Box B)$

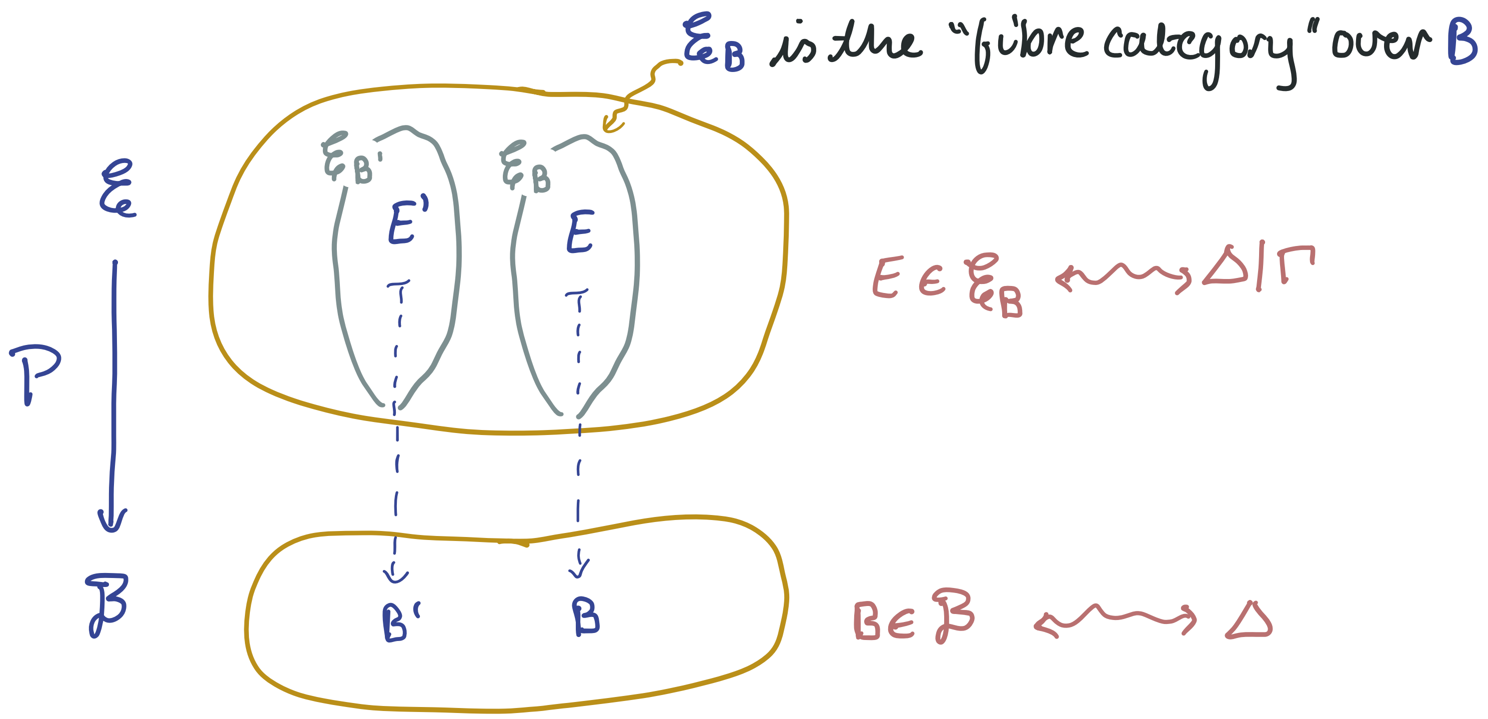
(T) $\Box A \to A$

(4) $\Box A \to \Box\Box A$

- Our strategy - zoom out

Modelling a split context type theory

For a context $\Delta | \Gamma$, want to capture the dependency of Γ on Δ.



$\mathcal{E}_B$ is the "fibre category" over B

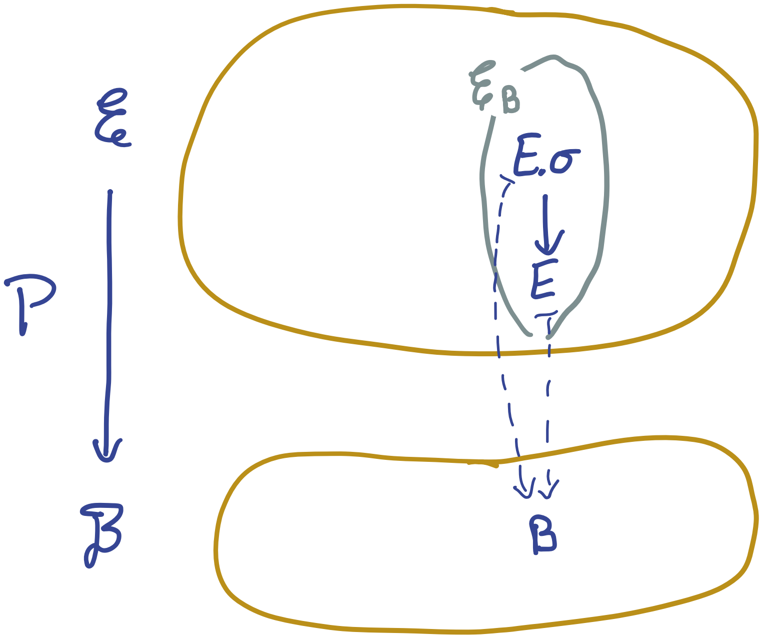
$E \in \mathcal{E}_B \leftrightsquigarrow \Delta | \Gamma$

$B \in \mathcal{B} \leftrightsquigarrow \Delta$

Modelling a split context type theory

Regular context extension:

$$\frac{\Delta \mid \Gamma \vdash \sigma \ \text{type}}{\Delta \mid \Gamma, \sigma \vdash}$$



$E.\sigma \leftrightsquigarrow \Delta \mid \Gamma, \sigma \vdash$

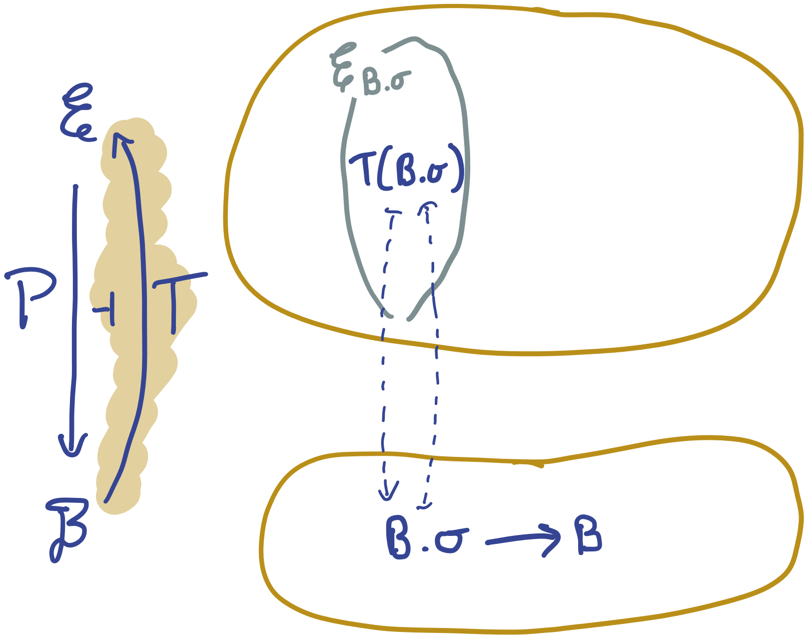
$E \leftrightsquigarrow \Delta \mid \Gamma \vdash$

$B \leftrightsquigarrow \Delta \vdash$

Modelling a split context type theory

Crisp context extension:

$$\frac{\Delta \mid \bullet \vdash \sigma \ \text{type}}{\Delta, \sigma \mid \bullet \vdash}$$



$T(B.\sigma) \leftrightsquigarrow \Delta, \sigma \mid \bullet \vdash$

$B \leftrightsquigarrow \Delta \vdash$

$B.\sigma \leftrightsquigarrow \Delta, \sigma \vdash$

Modelling crisp type theory

Idea Equip

(i) the base category, and
(ii) each fibre

with the structure to model a type theory.

universe of terms

universe of types

(e.g. two special objects $\tilde{\mathcal{U}}$ and $\mathcal{U}$, and map between them, $ty : \tilde{\mathcal{U}} \to \mathcal{U}$.

These universes should be related to each other so that

$\Delta \vdash_{\mathcal{B}} \sigma$ type and $\Delta | \cdot \vdash_{\mathcal{E}} \sigma$ type

are "the same".

What we've done

- Developed an abstract model of
  relativised, fibrewise "natural models"
  i.e. a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ + axioms
  $\rightsquigarrow$ helped us understand crisp type theory
- Zoomed back in to the concrete model

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[loop right, "b"] \arrow[d, bend left, "p_*"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Set} \arrow[u, bend left, "p^*"]
\end{tikzcd}$$

and shown this is an instance of the abstract model, where

$$\begin{tikzcd}
\mathcal{E} := & \hat{\mathbb{C}} \downarrow \hat{\mathbb{C}}_b \arrow[d, "\mathrm{cod}"] & \\
\mathcal{B} := & \hat{\mathbb{C}}_b & \text{full subcategory of } X \in \hat{\mathbb{C}} \text{ with } bX = X \arrow[l, squiggly]
\end{tikzcd}$$

n.b. $\hat{\mathbb{C}}_b \simeq \mathrm{Set}$

- extracted the internal (crisp!) type theory of $\hat{\mathbb{C}}\downarrow\hat{\mathbb{C}}_b$

$\hat{\mathbb{C}}\downarrow\hat{\mathbb{C}}_b$ $\rightsquigarrow$ **ingredients of a type theory $\mathcal{T}_{\hat{\mathbb{C}}\downarrow\hat{\mathbb{C}}_b}$**

object

$$\begin{tikzcd}
\Theta \arrow[d, "\Delta|\Gamma"] \\
\Gamma
\end{tikzcd}$$

$\rightsquigarrow$ context $\Delta|\Gamma$

map

$$\begin{tikzcd}
\Theta \arrow[r, "\alpha"] \arrow[d, "\Delta|\Gamma"'] & \mathcal{U} \arrow[d] \\
\Gamma \arrow[r] & 1
\end{tikzcd}$$

$\rightsquigarrow$ type-in-context

$\Delta|\Gamma \vdash \alpha$ is a type

diagram

$$\begin{tikzcd}
 & & & \tilde{\mathcal{U}} \arrow[dlll, bend right] \arrow[dl] \arrow[dd] \\
\Theta \arrow[rr, "\alpha"] \arrow[dd, "\Delta|\Gamma"'] & & \mathcal{U} \arrow[dd] & \\
 & & & 1 \arrow[dlll, bend right] \arrow[dl] \\
\Gamma \arrow[rr] & & 1 &
\end{tikzcd}$$

$\rightsquigarrow$ term-in-context

$\Delta|\Gamma \vdash a : \alpha$

"pullback" diagram

$$\begin{tikzcd}
 & \Theta.\alpha \arrow[rr] \arrow[dl] \arrow[dd] & & \tilde{\mathcal{U}} \arrow[dl] \arrow[dd] \\
\Theta \arrow[rr, "\alpha"] \arrow[dd, "\Delta|\Gamma"'] & & \mathcal{U} \arrow[dd] & \\
 & 1 \arrow[rr] \arrow[dl] & & 1 \arrow[dl] \\
\Gamma \arrow[rr] & & 1 &
\end{tikzcd}$$

$\rightsquigarrow$ context extension

$\Delta|\Gamma,\alpha \vdash q_\alpha : \alpha[p_\alpha]$

what we're doing

- developing Kripke-Joyal semantics in this setting
    - a general forcing condition, plus special cases for type formers

⇝ relates category-theoretic and type-theoretic descriptions e.g. for the universe of uniform fibrations

- Formalising the "functorial semantics"

- Formulating crisp Π-types in the model.

Thanks