Modal hyperdoctrine:

non-normal and higher-order extensions

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Plan

① Background

- algebraic semantics for propositional logics
- "algebraic semantics" for ... first order logics?
  ... higher order logics?

⇝ "hyperdoctrines"
  ↳ modal hyperdoctrines

② This work

Extensions to modal hyperdoctrine -

(i) hyperdoctrines for weak modal logics
(ii) relating modal hyperdoctrines to "standard" hyperdoctrines
(iii) higher-order hyperdoctrines for weak modal logics

Ingredients of propositional logic

- propositions $a, b, c, \ldots$ $\quad\longleftrightarrow\quad$ $a, b, c \in A$ a set
- entailment $a \vdash b$, with
  - (i) $a \vdash a$
  - (ii) if $a \vdash b$ and $b \vdash c$, then $a \vdash c$ $\quad\longleftrightarrow\quad$ $a \leq b$, $\leq$ a preorder on A

  + provable equivalence
  - (iii) if $a \vdash b$ and $b \vdash a$ then $a \equiv b$ $\quad\longleftrightarrow\quad$ $\leq$ a partial order on $A/\sim$
- connectives
  - $\top$
  - $\bot$
  - $a \wedge b$
  - $a \vee b$

  + axioms

$\longleftrightarrow$

$$\left.\begin{array}{l} \text{top element } \top \\ \text{bottom element } \bot \\ \text{meet operation } \wedge \\ \text{join operation } \vee \\ \quad + \text{ axioms} \end{array}\right\} \text{lattice}$$

Algebraisation

Propositional logic		Class of algebra
Intuitionistic	$\leftrightsquigarrow$	Heyting algebra
Classical	$\leftrightsquigarrow$	Boolean algebra
Modal S4	$\leftrightsquigarrow$	Interior algebra
etc.		

what about first order logic ?

Ingredients of first order logic

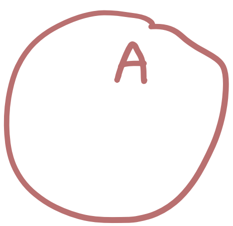
- propositions with variables

  φ with $\underbrace{\text{free variables } x_1, \dots, x_n}_{\text{"context"}}$

- quantifiers

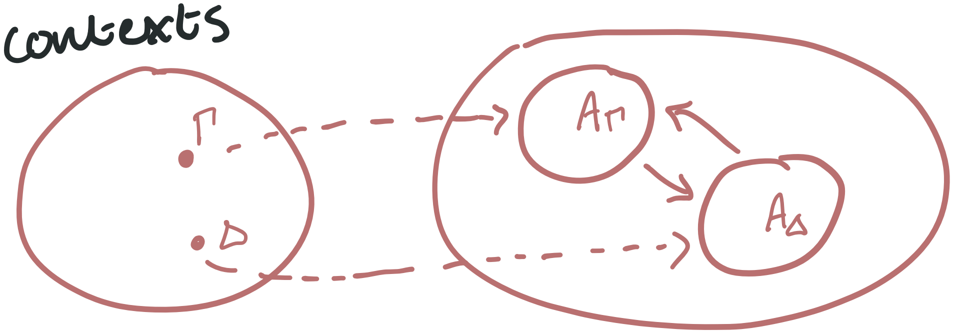
$$\begin{array}{cl} \varphi & \text{a formula w/ free variables } x_1, x_2, \dots, x_n \\ \hline\hline \forall x_1. \varphi & \text{a formula w/ free variables } x_2, \dots, x_n \end{array}$$

Propositional logic vs. **First order logic**



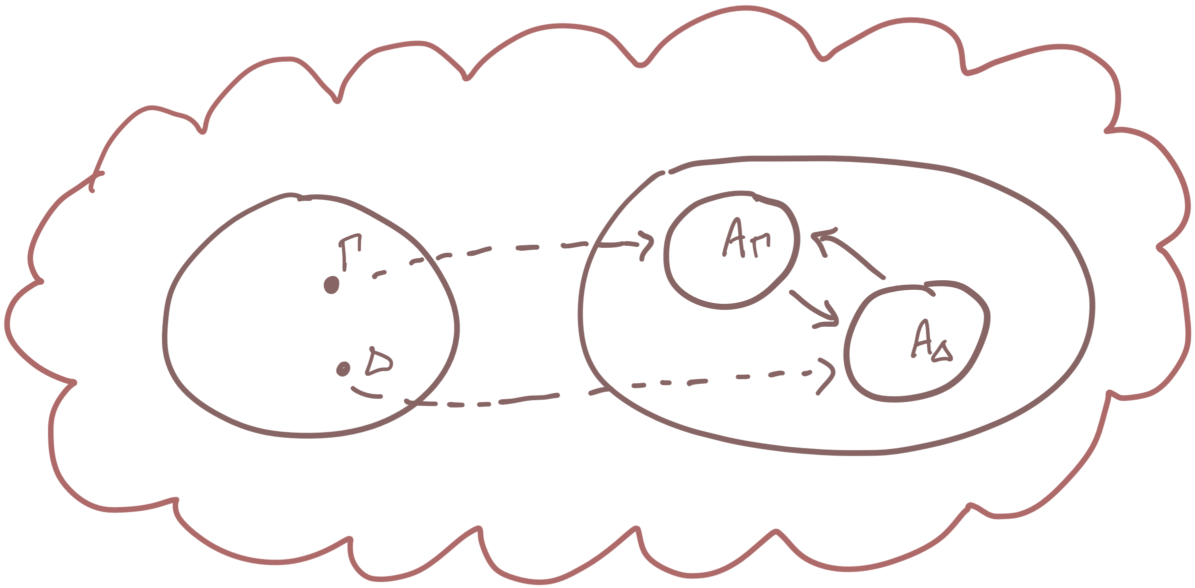
A a Heyting/Boolean/... algebra

contexts



$A_\Gamma, A_\Delta, \dots$ Heyting/Boolean/... algebras

Formalising our picture



- This picture can be formalised with category theory
  $\rightsquigarrow$ "categorical semantics", but also an algebraisation of predicate logic
- First, we have to add a "type structure" to our syntax...

Typed first order language

Main idea: every formula is a formula in a (typed) context

$\varphi\ [\Gamma]$ where $\Gamma = x_1 : A_1, \ldots, x_n : A_n$

LOGIC

axioms $(\forall x.\varphi)[x_1,\ldots,x_n] \supset \varphi\ [x:A,\Gamma]$

inference rules $\dfrac{\varphi\ [\Gamma] \quad \varphi \supset \psi\ [\Gamma]}{\psi\ [\Gamma]}$

compound formulae $\varphi \supset \psi\ [\Gamma]$, $\forall x.\varphi\ [\Gamma]$, ...

$$\frac{t_1 : A_1\,[\Gamma], \ldots, t_n : A_n\,[\Gamma] \qquad R \subseteq A_1, \ldots, A_n}{R(t_1, \ldots, t_n)\ [\Gamma]}$$ is an atomic formula

TYPE THEORY

terms in context $t : A\ [\Gamma]$

predicate symbols $R \subseteq A_1, \ldots, A_n$

types $A_1, A_2, \ldots$ function symbols $F : A_1, \ldots, A_n \to B$

Hyperdoctrines

- F. W. Lawvere, 1969, "Adjointness in Foundations"

Let $\mathcal{C}$ be a category with finite products. type structure

A **Lawvere hyperdoctrine** is a functor

$$P : \mathcal{C}^{op} \longrightarrow HA$$

$$X \longmapsto P(X)$$ ⟵ Heyting algebra

$$\begin{array}{ccc} X & & P(X) \\ f \downarrow & \longmapsto & \uparrow P(f) \\ X' & & P(Y) \end{array}$$ ⟵ Finite meet preserving function

such that

(i) for all projection maps π,

$$\begin{tikzcd} X \times Y \arrow[d, "\pi"] \\ Y \end{tikzcd} \longmapsto \begin{tikzcd} P(X \times Y) \arrow[d, bend right, "\exists_\pi"'] \arrow[d, bend left, "\forall_\pi"] \\ P(Y) \arrow[u, "P(\pi)"] \end{tikzcd}$$

$P(\pi)$ has a right adjoint "$\forall_\pi$" and a left adjoint "$\exists_\pi$"

Illustration in the "syntactic hyperdoctrine":

For $[x:A,\ y:B] \xrightarrow{\pi} [y:B]$, we have

$$\begin{tikzcd}[column sep=large]
\varphi \quad [y:B] \arrow[rr, mapsto, dashed, "\text{"weakening"}"'] & & \varphi \quad [x:A, y:B] \\
P([y:B]) \arrow[rr, "P(\pi)"] & & P([x:A, y:B]) \arrow[ll, bend left, "\forall_\pi"] \\
\forall x.\psi \quad [y:B] & & \psi \quad [x:A, y:B] \arrow[ll, mapsto, dashed, bend left]
\end{tikzcd}$$

and the adjunction means

$$\frac{\varphi \vdash \forall x.\psi \qquad [y:B]}{\varphi \vdash \psi \qquad [x:A, y:B]}$$

Hyperdoctrines

- F. W. Lawvere, 1969, "Adjointness in Foundations"

A **Lawvere hyperdoctrine** is a functor $P : \mathcal{C}^{op} \longrightarrow HA$ such that

(i) for all projection maps π,

$$
\begin{array}{c} X \times Y \\ \pi \downarrow \\ Y \end{array}
\quad \longmapsto \quad
\begin{tikzcd}
P(X \times Y) \arrow[d, bend right=60, "\exists_\pi"'] \arrow[d, bend left=60, "\forall_\pi"] \\
P(Y) \arrow[u, "P(\pi)"']
\end{tikzcd}
$$

$P(\pi)$ has a right adjoint "$\forall_\pi$" and a left adjoint "$\exists_\pi$"

(ii) all $\forall_\pi$ and $\exists_\pi$ satisfy "Beck-Chevalley" conditions

quantifiers commute w/ substitution

(iii) all $\exists_\pi$ satisfy "Frobenius Reciprocity"

Example Syntactic hyperdoctrine

Ctx is the category of contexts* $\Gamma = [x_1 : A_1, \dots, x_n : A_n]$ and arrows between contexts, given by lists of terms**

$$
\begin{tikzcd}
\Gamma = [x_1 : A_1, \dots, x_n : A_n] \arrow[d, "{[t_1 : B_1[\Gamma], \dots, t_m : B_m[\Gamma]]}"] \\
\Delta = [y_1 : B_1, \dots, y_m : B_m]
\end{tikzcd}
$$

Then $P : \mathbf{Ctx}^{op} \longrightarrow HA$

$\Gamma \longmapsto \{\varphi \mid \varphi \text{ a formula in context } \Gamma\}/\sim$

$$
\begin{tikzcd}
\Gamma \arrow[d, "{[t_1, \dots, t_m]}"'] & \longmapsto & P(\Gamma) & \psi[t_1/y_1, \dots, t_m/y_m] \; [\Gamma] \\
\Delta & & P(\Delta) \arrow[u] & \psi \; [\Delta] \arrow[u, mapsto]
\end{tikzcd}
$$

* up to α-equivalence ** up to term equivalence

Example Syntactic hyperdoctrine (cont.)

Right adjoints to images of projection maps:

$$P : \underline{Ctx}^{op} \longrightarrow HA$$

$$\begin{tikzcd}
\Gamma \times \Delta \arrow[d, "\pi"'] & & P(\Gamma \times \Delta) \arrow[d, bend left] & \varphi \quad [\Gamma, \Delta] \arrow[d, mapsto] \\
\Gamma & \longmapsto & P(\Gamma) \arrow[u] & \forall y_1 \ldots \forall y_m . \varphi \quad [\Gamma]
\end{tikzcd}$$

How does this give us a semantics?

1) Define an interpretation in a hyperdoctrine $P : \mathcal{C}^{op} \to HA$

- the type theory is interpreted (inductively) in $\mathcal{C}$

  e.g. A a type $\rightsquigarrow$ $[\![A]\!]$ an object in $\mathcal{C}$

  $\Gamma = x_1 : A_1, \ldots, x_n : A_n \rightsquigarrow [\![\Gamma]\!] = [\![A_1]\!] \times \ldots \times [\![A_n]\!]$ in $\mathcal{C}$

- the logic is interpreted (inductively) in HA

  e.g. $\varphi \wedge \psi\ [\Gamma] \rightsquigarrow [\![\varphi\ [\Gamma]]\!] \wedge_{P([\![\Gamma]\!])} [\![\psi\ [\Gamma]]\!]$

- quantifiers are interpreted as adjoints

  e.g. $\forall x\, \varphi\ [\Gamma] \rightsquigarrow \forall_{\pi}([\![\phi\ [x : A, \Gamma]]\!])$

How does this give us a semantics?

2) Define satisfaction of a formula in an interpretation

$\varphi\,[\Gamma]$ is satisfied in $(\llbracket - \rrbracket, P)$ $\iff$ $\llbracket \varphi \rrbracket = \top_{P(\llbracket \Gamma \rrbracket)}$

3) Prove soundness and completeness

$\varphi\,[\Gamma]$ provable $\iff$ $\varphi\,[\Gamma]$ is satisfied in any $(\llbracket - \rrbracket, P)$

Modal logic - semantics

- Kripke semantics are great for propositional modal logics

⚠ they don't automatically extend to first-order cases

"Alternative" semantics for quantified modal logics:

- Category-theoretic semantics used by
  - Ghilardi (e.g. 1991, 2001), Ghilardi and Meloni (1988, 1990, 1991)
    ⇝ modal hyperdoctrines
    = hyperdoctrines valued in **Int**
  - Awodey, Kishida and Kotzsch (2014, 2016)
    ⇝ Topos-theoretic hyperdoctrine semantics
    for higher order modal logic (intuitionistic S4)

Part two: going further with modal hyperdoctrines*

*classical

① Non-normal modal hyperdoctrine

Idea: Give a more general presentation of modal hyperdoctrines, including modal logics weaker than S4

- Define a modal hyperdoctrine to be an MA-valued hyperdoctrine

$$\mathrm{obj}(MA) \qquad (\underbrace{A, \top_A, \bot_A, \wedge_A, \vee_A, \neg_A}_{\text{Boolean algebra}}, \underbrace{\Box_A : A \to A}_{\text{operator w/ zero or more conditions}})$$

		LOGIC	ALGEBRA
S4, normal	E	$\Diamond\varphi \supset \neg\Box\neg\varphi$	$\Diamond_A(x) := \neg_A \Box_A(\neg_A x)$
	M	$\Box(x \wedge y) \supset \Box x \wedge \Box y$	$\Box_A(x \wedge_A y) \leq \Box_A x \wedge_A \Box_A y$
	C	$\Box x \wedge \Box y \supset \Box(x \wedge y)$	$\Box_A x \wedge \Box_A y \leq \Box_A(x \wedge_A y)$
	N	$\Box\top \supset \top$	$\Box_A \top_A = \top_A$
	T	$\Box x \supset x$	$\Box_A x \leq x$
	4	$\Box x \supset \Box\Box x$	$\Box_A x \leq \Box_A \Box_A x$

② Translation theorem

Idea: relate modal hyperdoctrines to standard notion of hyperdoctrine

modal hyperdoctrine

Gödel-McKinsey-Tarski translation

$A \longmapsto \{a \in A \mid \Box a = a\}$

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "P"] \arrow[rr, bend right, "\mathrm{Fix}_\Box \circ P"'] & MA \arrow[r, "\mathrm{Fix}_\Box"] & HA
\end{tikzcd}$$

is an intuitionistic hyperdoctrine

⚠ only works for modal logics S4 and stronger

③ Higher-order modal hyperdoctrine

Idea: Higher-order hyperdoctrines (aka "triposes"*) can easily be adapted to modal logic

- Higher-order syntax
  a) add rules for $\to$, $\times$ types to the term calculus
  b) add a distinguished type Prop to the signature
  $\rightsquigarrow$ quantification over predicates

- Higher-order hyperdoctrine $P : \mathcal{C}^{op} \to MA$
  a) $\mathcal{C}$ is cartesian closed
  b) $\mathcal{C}$ has a "truth value object" Ω with

$$P(C) \cong \mathrm{Hom}_{\mathcal{C}}(C, \Omega)$$

* Hyland, Johnstone, Pitts, "Tripos Theory", 1980

Conclusions

- Hyperdoctrines are algebras indexed by a category, with adjoints to the images of certain maps, where

  the category $\leftrightsquigarrow$ type structure
  the algebras $\leftrightsquigarrow$ logic structure
  adjoints $\leftrightsquigarrow$ quantifiers

- When you have the right concept for the job, the job is straightforward

- next stop: coalgebraic logic?