Modal Hyperdoctrine: Higher-Order and Non-normal Extensions

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Abstract. Lawvere hyperdoctrines give categorical semantics for intuitionistic predicate logic but are flexible enough to be applied to other logics and extended to higher-order systems. We return to Ghilardi's hyperdoctrine semantics for first-order modal logic [3] and extend it in two directions—to weaker, non-normal modal logics and to higher-order modal logics. We also relate **S4** modal hyperdoctrines to intuitionistic hyperdoctrines via a hyperdoctrinal version of the Gödel-McKinsey-Tarski translation. This work is intended to complement the other categorical semantics that have been developed for quantified modal logic, and may also be regarded as first steps to extend coalgebraic modal logic to first-order and higher-order settings via hyperdoctrines.

Keywords: Categorical logic · Modal logic · Higher-order logic · Hyperdoctrines

1 Introduction

Moving from propositional modal logic to quantified modal logic is less straightforward than one might hope. For example, traditional Kripke semantics do not automatically extend to the first-order case, with several instances of well-motivated but incomplete extensions of Kripke-complete propositional logics [5]. Turning to alternative semantics, category-theoretic methods have been used extensively by Ghilardi and Meloni [7–11] for mathematical and philosophical investigations of quantified modal logic beyond the reach of Kripke semantics.

Amongst the category-theoretic tools deployed are Lawvere's *hyperdoctrines* [15]. Hyperdoctrines provide semantics for first-order logics that reduce to familiar algebraic semantics on the propositional level. Originally conceived for intuitionistic predicate logic, they are flexible enough to be applied to other logics and extended to higher-order systems. Hyperdoctrine semantics for first-order normal modal logics are presented in [3], where they are used by Ghilardi as a unifying tool for studying other non-Kripkean modal semantics, while Awodey, Kishida and Kotzsch [2,13] provide topos-theoretic hyperdoctrine semantics for higher-order modal logic based on intuitionistic **S4**.

We make three contributions to modal hyperdoctrine. The first is a very general presentation. Ghilardi's presentation in [3] concerns a single-sorted typed

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language and a base propositional modal logic of **S4**, while advising that it is straightforward to generalise. We follow this guidance to present modal hyperdoctrine semantics for a many-sorted typed language and a base propositional modal logic of the weaker *non-normal* class. To this end, we introduce hyperdoctrines in Sect. 2, present the syntax of the modal logic in Sect. 3, and give the semantics in Sect. 4. The second contribution is to connect modal hyperdoctrines—in the case of **S4** modal logics—to intuitionstic hyperdoctrines via a translation theorem (Sect. 4.3). The third is to define higher-order modal hyperdoctrines for non-normal modal logics and prove their soundness and completeness (Sect. 5). This complements the aforementioned work of Awodey et al., in which the topos-theoretic nature of their semantics prohibits generalising to bases weaker than **S4**. In Sect. 6 we conclude with future directions.

2 Hyperdoctrine Semantics

In this section, we define a hyperdoctrine and consider when a logic has semantics in a hyperdoctrine. Many decisions are made in choosing a quantified modal logic, from the base propositional logic to the interaction between the modal operators and quantifiers. We take the perspective: if we want sound and complete hyperdoctrine semantics for modal logic, what does it require of our logic?

Lawvere hyperdoctrines are fibred algebras indexed by categories, where the algebras represent the propositional logic and the indexing category provides a type structure. Let $\mathbf{C}$ be a category with finite products and $\mathbf{HA}$ be the category of Heyting algebras and finite meet preserving functions between them. A hyperdoctrine is a functor

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$$

capturing quantification by the following requirements. For any projection $\pi : X \times Y \to Y$ in $\mathbf{C}$, the image $P(\pi) : P(Y) \to P(X \times Y)$ has right and left adjoints

$$\forall_\pi : P(X \times Y) \to P(Y) \text{ and } \exists_\pi : P(X \times Y) \to P(Y).$$

These adjoints satisfy corresponding Beck-Chevalley conditions: for $\forall_\pi$, this says that the following diagram commutes for any $f : Z \to Y$ in $\mathbf{C}$, where $\pi' : X \times Z \to Z$ is a projection[1]:

$$\begin{tikzcd} P(X \times Y) \arrow[r, "\forall_\pi"] \arrow[d, "P(\mathsf{id}_X \times f)"'] & P(Y) \arrow[d, "P(f)"] \\ P(X \times Z) \arrow[r, "\forall_{\pi'}"'] & P(Z) \end{tikzcd} \tag{1}$$

The indexing category $\mathbf{C}$ represents a type structure that acts as a domain of reasoning for the logic. In this way, hyperdoctrines adopt the view that "a logic

[1] The left adjoint must also satisfy the *Frobenius reciprocity condition*, omitted here as we are only concerned with classical logic, in which the quantifiers are interdefined.

is always a logic over a type theory" [12]. This is more natural from a category-theoretic perspective and subsumes untyped logics via reduction to a single type.

The restrictions placed on the syntax of our logic if we wish to equip it with hyperdoctrine semantics are as follows. The syntax is a typed version, built on top of a type signature and term calculus, detailed in Sect. 3. The functoriality of P means that substitution commutes with all of the logical connectives. This is clear when we consider the syntactic hyperdoctrine in Sect. 4, where we see that in order for the image of a map in the base category to be an algebra homomorphism, it is necessary that substitution commutes with the propositional connectives. Considering the syntactic hyperdoctrine also demonstrates that the Beck-Chevalley condition corresponds logically to the quantifiers commuting with substitution, and so we also require this of our syntax.

3 Typed First-Order Non-normal Modal Logic

Non-normal modal logics are a particularly weak class of modal logics, as distinct from *normal* modal logics such as **K** and **S4**. In this section, we present a typed version of first-order non-normal modal logics, following [1] for the logic and [19] for the typing. The resulting system is essentially a multi-sorted, non-normal version of the single-sorted, normal logic in [3].

3.1 Term Calculus

The logic is built on a typed (many-sorted) *signature* Σ, consisting of *type symbols* σ, *function symbols* $F : \sigma_1, \ldots, \sigma_n \to \tau$ and *relation (predicate) symbols* $R \subseteq \sigma_1, \ldots, \sigma_n$. For each type σ there are *variables* $x, y, z, \ldots$, and the formal expression $x : \sigma$ is a *type judgement* expressing that x is a variable of type σ. A *context* is a finite list of type judgements $x_1 : \sigma_1, \ldots, x_n : \sigma_n$, denoted by Γ.

On top of the signature is a *term calculus*. The basic term calculus consists of *terms-in-context*, which are judgements $M : \sigma\ [\Gamma]$, expressing that M is a well-formed term of type σ in context Γ. The well-formed terms-in-context in the basic term calculus are inductively generated by the following rules:

- $x : \sigma\ [\Gamma, x : \sigma, \Gamma']$ is a term;
- if $F : \sigma_1, \ldots, \sigma_n \to \tau$ is a function symbol and $M_1 : \sigma_1\ [\Gamma], \ldots, M_n : \sigma_n\ [\Gamma]$ (abbreviated $\vec{M} : \vec{\sigma}$) are terms, then $F(M_1, \ldots, M_n) : \tau\ [\Gamma]$ is a term.

The meta-theoretic operation of *substitution over a term* of a term for a variable is defined by induction on the structure of an untyped term N:

- if $N = x_i$ then $N[\vec{M}/\vec{x}] = M_i$;
- if $N = F(N_1, \ldots, N_n)$ then $N[\vec{M}/\vec{x}] = F(N_1[\vec{M}/\vec{x}], \ldots, N_n[\vec{M}/\vec{x}])$.

A *formula-in-context* is a judgement $\phi\ [\Gamma]$ expressing that ϕ is a well-formed formula in context Γ. For each relation symbol $R \subseteq \sigma_1, \ldots, \sigma_n$, if $M_1 : \sigma_1\ [\Gamma], \ldots, M_n : \sigma_n[\Gamma]$ are terms, then $R(M_1, \ldots, M_n)\ [\Gamma]$ is an *atomic formula*. *Compound formulae* are built from the atomic formulae and the constant $\perp$ with the rules:

- $\bot\ [\Gamma]$ is a formula;
- if $\phi\ [\Gamma]$ and $\psi\ [\Gamma]$ are formulae then $\phi \supset \psi\ [\Gamma]$ is a formula;
- if $\phi\ [x : \sigma, \Gamma]$ is a formula then $\forall x \phi\ [\Gamma]$ is a formula;
- if $\phi\ [\Gamma]$ is a formula then $\Box\phi\ [\Gamma]$ is a formula.

The remaining connectives are treated as abbreviations in the usual manner; this includes equivalence of formulae $\phi \supset\subset \psi$, abbreviating $\phi \supset \psi \wedge \psi \subset \phi$.

If $\phi\ [\Gamma]$ is a formula with $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ and $M_1 : \sigma_1\ [\Gamma'], \ldots, M_n : \sigma_n\ [\Gamma']$ are terms, we want to define a formula $\phi[\vec{M}/\vec{x}]\ [\Gamma']$, where every instance of the variable x_i is replaced by the term M_i, for every i. Since every formula is built in a unique way from atomic subformulae and the rules for forming compound formulae, substitution into a formula is defined on these subformulae as follows. Substitution over atomic formulae:

$$R(N_1, \ldots, N_n)[\vec{M}/\vec{x}]\ [\Gamma'] \coloneqq R(N_1[\vec{M}/\vec{x}], \ldots, N_n[\vec{M}/\vec{x}])\ [\Gamma']$$

Substitution on subformulae (where x_{m+1} is a fresh variable):

- $\bot[\vec{M}/\vec{x}]\ [\Gamma'] \coloneqq \bot\ [\Gamma']$
- $(\phi_1 \supset \phi_2)[\vec{M}/\vec{x}]\ [\Gamma'] \coloneqq (\phi_1[\vec{M}/\vec{x}]) \supset (\phi_2[\vec{M}/\vec{x}])\ [\Gamma']$
- $(\forall x_{n+1}\psi)[\vec{M}/\vec{x}]\ [\Gamma'] \coloneqq \forall x_{m+1}(\psi[\vec{M}/\vec{x}, x_{m+1}/x_{n+1}])\ [\Gamma']$
- $(\Box\psi)[\vec{M}/\vec{x}]\ [\Gamma'] \coloneqq \Box(\psi[\vec{M}/\vec{x}])\ [\Gamma']$

3.2 Logical Calculus

A Hilbert-style system for (typed) non-normal propositional modal logics is given by any axiomatisation of propositional logic, plus the rules and axiom schema

$$\frac{\phi \supset\subset \psi\ [\Gamma]}{\Box\phi \supset\subset \Box\psi\ [\Gamma]}(\mathrm{RE}) \qquad \frac{\phi\ [\Gamma] \quad \phi \supset \psi\ [\Gamma]}{\psi\ [\Gamma]}(\mathrm{MP}) \qquad \Diamond\phi \supset\subset \neg\Box\neg\phi\ [\Gamma]\ (\mathrm{E})$$

and zero or more of the following axiom schemata:

$$\Box(\phi \wedge \psi) \supset (\Box\phi \wedge \Box\psi) \quad [\Gamma] \tag{M}$$

$$(\Box\phi \wedge \Box\psi) \supset \Box(\phi \wedge \psi) \quad [\Gamma] \tag{C}$$

$$\Box\top \quad [\Gamma] \tag{N}$$

The smallest non-normal propositional modal logic is called **E**; the non-normal extensions are denoted by $\mathbf{E}_X$, where X is a subset of $\{\mathbf{M}, \mathbf{N}, \mathbf{C}\}$ and $\mathbf{E}_X$ is the smallest system containing every instance of the axiom schemata in X. The system **EMCN** is equivalent to the smallest normal modal logic **K**. The system **S4** is **K** plus the schemata $\Box\phi \supset \phi\,[\Gamma]$ (T) and $\Box\phi \supset \Box\Box\phi\,[\Gamma]$ (4).

To extend any propositional non-normal modal logic **S** to a (typed) first-order logic **TFOL** + **S**, we add the following axiom schema and rules:[2]

$$(\forall x\phi)[x_1,\dots,x_n] \supset \phi \quad [x:\sigma,\Gamma] \qquad (\forall\text{-}Elim)$$

$$\frac{\phi[x_1,\dots,x_n] \supset \psi \quad [x:\sigma,\Gamma]}{\phi \supset \forall x\psi \quad [\Gamma]}\ (\forall\text{-}Intro) \qquad \frac{\phi \quad [\Gamma]}{\phi[\vec{M}/\vec{x}] \quad [\Gamma']}\ (Inst)$$

where $\Gamma = x_1:\sigma_1,\dots,x_n:\sigma_n$ and $\vec{M}$ abbreviates the terms $M_1:\sigma_1\ [\Gamma']$, ..., $M_n:\sigma_n\ [\Gamma']$. The formula $\phi[x_1,\dots,x_n]$ evidentiates the free variables of ϕ, since the rule requires that x is not free in ϕ. It differs from the formula $\phi\ [\Gamma]$ by the renaming of bound variables.

A *derivation* of a formula $\phi\ [\Gamma]$ is a finite sequence of formulae $\phi_1\ [\Gamma_1]$, $\phi_2\ [\Gamma_2]$, ..., $\phi_n\ [\Gamma_n]$ such that each formula is either an instance of an axiom schema or follows from earlier formulae by one of the rules of inference. A formula $\phi\ [\Gamma]$ is said to be *derivable* in the axiom system **TFOL** + **S** if there exists a derivation of $\phi\ [\Gamma]$ in this axiom system, denoted $\vdash_{\mathbf{TFOL}\,+\,\mathbf{S}} \phi\ [\Gamma]$.

4 Hyperdoctrine Semantics for TFOL + $\mathbf{E}_{\boldsymbol{X}}$

Before defining a modal hyperdoctrine, we present the standard algebraic semantics for modal logic, to which the hyperdoctrine semantics reduce on the propositional level. Algebraic semantics for modal logic **S4** were developed by McKinsey and Tarski, extended to normal modal logics in [16], and even weaker modal logics in [6]. We adopt this last, most general, definition of modal algebra.

Definition 1. *A* modal algebra A *is a Boolean algebra* $(A, \wedge_A, \vee_A, \neg_A, \top_A, \bot_A)$ *together with a unary operator* $\Box_A$ *satisfying zero or more conditions, such as:*

$$\Box_A(x \wedge_A y) \le \Box_A(x) \wedge_A \Box_A(y) \qquad (\mathrm{M}_A)$$

$$\Box_A(x) \wedge_A \Box_A(y) \le \Box_A(x \wedge_A y) \qquad (\mathrm{C}_A)$$

$$\Box_A(\top_A) = \top_A \qquad (\mathrm{N}_A)$$

$$\Box_A(x) \le x \qquad (\mathrm{T}_A)$$

$$\Box_A(x) \le \Box_A\Box_A(x) \qquad (4_A)$$

There are secondary operations $x \supset_A y := \neg_A x \vee_A y$ *and* $\Diamond_A(x) := \neg_A\Box_A(\neg_A x)$.

We use the same notation for the operations on the algebra as for the logical connectives, to highlight their correspondence. The algebraic operations are subscripted with the underlying set when it is helpful to have a reminder that we are in the algebraic setting. A poset structure is inherited from the Boolean algebra, given by the order $x \le y$ if and only if $x \wedge_A y = x$. Modal algebras and finite meet preserving functions between them form the category **MA**.

[2] This axiomatisation deviates from [1], instead following [3] in taking two separate principles of *replacement*—corresponding to the *Instantiation* rule—and *agreement*—corresponding to the ∀ -*Introduction* rule—to more readily accommodate the proofs.

Possible conditions on $\Box_A$ correspond to axiom schemata of the logical calculus to be captured. In the proofs that follow, we only specify the strength of modal algebra to which the category **MA** refers when necessary. Since we are concerned with the level of predicates, most proofs operate independently of the specific axioms satisfied by the modal operator.

4.1 Modal Hyperdoctrine Semantics

In this section we adapt the definition of Lawvere hyperdoctrine from intuitionistic logic to modal logic, define *interpretation* in a modal hyperdoctrine, and prove that this gives sound and complete semantics for $\mathbf{TFOL} + \mathbf{E}_X$.

Definition 2. *Let* $\mathbf{C}$ *be a category with finite products. A* modal hyperdoctrine *is a contravariant functor* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MA}$ *such that for any projection* $\pi : X \times Y \to Y$ *in* $\mathbf{C}$*,* $P(\pi) : P(Y) \to P(X \times Y)$ *has a right adjoint satisfying the Beck-Chevalley condition (1).*

Since our modal logic is classical, our definition of modal hyperdoctrine does not treat the existential quantifier independently.

Definition 3. *Fix a modal hyperdoctrine* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MA}$*. An* interpretation $[\![\text{-}]\!]$ *of* $\mathbf{TFOL} + \mathbf{E}_X$ *in* P *consists of the following:*

- *assignment of an object* $[\![\sigma]\!]$ *in* $\mathbf{C}$ *to each basic type* σ *in* $\mathbf{TFOL} + \mathbf{E}_X$*;*
- *assignment of an arrow* $[\![F]\!] : [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!] \to [\![\tau]\!]$ *in* $\mathbf{C}$ *to each function symbol* $F : \sigma_1, \ldots, \sigma_n \to \tau$ *in* $\mathbf{TFOL} + \mathbf{E}_X$*;*
- *assignment of an element* $[\![R\ [\Gamma]]\!]$ *in the modal algebra* $P([\![\Gamma]\!])$ *to each typed predicate symbol* $R\ [\Gamma]$ *in* $\mathbf{TFOL} + \mathbf{E}_X$*; if the context* Γ *is* $x_1 : \sigma_1, ..., x_n : \sigma_n$*, then* $[\![\Gamma]\!]$ *denotes* $[\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!]$*.*

The interpretation of a term is defined by induction on its derivation, as follows:

- $[\![x : \sigma\ [\Gamma, x : \sigma, \Gamma']]\!]$ *is defined as the following projection in* $\mathbf{C}$*:*
$$\pi : [\![\Gamma]\!] \times [\![\sigma]\!] \times [\![\Gamma']\!] \to [\![\sigma]\!];$$
- $[\![F(M_1, \ldots, M_n) : \tau\ [\Gamma]]\!] := [\![F]\!] \circ \langle [\![M_1 : \sigma_1\ [\Gamma]]\!], \ldots, [\![M_n : \sigma_n\ [\Gamma]]\!] \rangle$.

Formulae are interpreted inductively in the following manner:

- $[\![R(M_1, \ldots, M_n)\ [\Gamma]]\!] := P(\langle [\![M_1 : \sigma_1\ [\Gamma]]\!], \ldots, [\![M_n : \sigma_n\ [\Gamma]]\!] \rangle)([\![R]\!])$.
- *For the propositional connectives:*

$$\begin{aligned}
[\![\phi \wedge \psi\ [\Gamma]]\!] &:= [\![\phi\ [\Gamma]]\!] \wedge_{P([\![\Gamma]\!])} [\![\psi\ [\Gamma]]\!] \\
[\![\phi \vee \psi\ [\Gamma]]\!] &:= [\![\phi\ [\Gamma]]\!] \vee_{P([\![\Gamma]\!])} [\![\psi\ [\Gamma]]\!] \\
[\![\phi \supset \psi\ [\Gamma]]\!] &:= [\![\phi\ [\Gamma]]\!] \supset_{P([\![\Gamma]\!])} [\![\psi\ [\Gamma]]\!] \\
[\![\neg\phi\ [\Gamma]]\!] &:= \neg_{P([\![\Gamma]\!])} [\![\phi\ [\Gamma]]\!] & [\![\bot\ [\Gamma]]\!] &:= \bot_{P([\![\Gamma]\!])} \\
[\![\Box\phi\ [\Gamma]]\!] &:= \Box_{P([\![\Gamma]\!])}([\![\phi\ [\Gamma]]\!]) & [\![\top[\Gamma]]\!] &:= \top_{P([\![\Gamma]\!])}
\end{aligned}$$

– *For the quantifiers:*

$$[\![\forall x\phi\ [\Gamma]]\!] := \forall_\pi([\![\phi\ [x:\sigma,\Gamma]]\!]) \quad \textit{and} \quad [\![\exists x\phi\ [\Gamma]]\!] := \exists_\pi([\![\phi\ [x:\sigma,\Gamma]]\!])$$

where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ *is a projection in* $\mathbf{C}$.

For a formula $\phi\ [\Gamma]$, where $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, and terms $M_1 : \sigma_1\ [\Gamma'], \ldots, M_n : \sigma_n\ [\Gamma']$, the interpretation of substitution by $\vec{M}$ is:

$$[\![\phi[\vec{M}/\vec{x}]\ [\Gamma']]\!] = P(\langle [\![M_1 : \sigma_1\ [\Gamma']]\!], \ldots, [\![M_n : \sigma_n\ [\Gamma']]\!]\rangle)([\![\phi\ [\Gamma]]\!]).$$

This can be proved by induction on the structure of ϕ. Weakening of the context of a formula $\phi\ [\Gamma]$ to the context $x : \sigma, \Gamma$ is the following special case:

$$[\![\phi\ [x:\sigma,\Gamma]]\!] = P(\pi)([\![\phi\ [\Gamma]]\!])$$

where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ is a projection map.

Definition 4. *A formula* $\phi\ [\Gamma]$ *is* satisfied *in an interpretation* $[\![\text{-}]\!]$ *in a modal hyperdoctrine* P *if and only if* $[\![\phi]\!] = \top_{P([\![\Gamma]\!])}$.

Since $a \leq_A \top_A$ holds for every element a in a Boolean algebra A, showing the satisfiability of $\phi\ [\Gamma]$ amounts to showing $\top_{P([\![\Gamma]\!])} \leq [\![\phi]\!]$. Note that the definition of satisfaction here differs from that in [19], which is concerned with the satisfiability of sequents rather than formulae.

4.2 Soundness and Completeness

We proceed by proving the soundness and completeness of $\mathbf{TFOL} + \mathbf{E}_X$ with respect to the modal hyperdoctrine semantics. We make use of an equivalent condition for the satisfaction of an implication $\phi \supset \psi\ [\Gamma]$ in an interpretation:

$$\top \leq [\![\phi \supset \psi\ [\Gamma]]\!] \text{ if and only if } [\![\phi\ [\Gamma]]\!] \leq [\![\psi\ [\Gamma]]\!]. \tag{2}$$

This follows from the fact that in a Boolean algebra, the pair of functions $\text{-} \wedge x : A \to A$ and $x \supset \text{-} : A \to A$ determine an adjunction, that is, for all $y, z \in A$, $z \leq x \supset y$ if and only if $z \wedge x \leq y$. Letting $z = \top$ and using the fact that $\top \wedge x = x$, we have $\top \leq x \supset y$ if and only if $x \leq y$.

We will also use the following bijection, coming from the adjointness condition on the universal quantifier:

$$\frac{P(\pi)(A) \leq_{P(X \times Y)} B}{A \leq_{P(Y)} \forall_\pi(B)} \tag{3}$$

The following soundness proof is with respect to the systems $\mathbf{TFOL} + \mathbf{E}_X$, but we note that the proof applies to other systems $\mathbf{TFOL} + \mathbf{S}$, provided we strengthen the conditions on the modal operator in correspondence with the axiom schemata of $\mathbf{S}$. This generality is possible given how the predicate and propositional components interact in the semantics, that is, the structure on the predicate part governs the interaction between the modal algebras, while preserving their internal structure.

Proposition 1. *If ϕ $[\Gamma]$ has a derivation in* $\mathbf{TFOL} + \mathbf{E}_X$*, then it is satisfied in any interpretation in any modal hyperdoctrine.*

Proof. See Appendix A.

Towards proving completeness, we now define the syntactic hyperdoctrine of $\mathbf{TFOL} + \mathbf{S}$. For the base category $\mathbf{C}$, let the objects be contexts Γ up to α-equivalence (renaming of variables). This is equivalent to taking as objects lists of types $\sigma_1, \ldots, \sigma_n$, rather than a list of variable-type pairs. A context morphism from $\sigma_1, \ldots, \sigma_n$ to $\Gamma' = \tau_1, \ldots, \tau_m$ is given by a list of terms $t_1 : \tau_1[\Gamma], \ldots, t_m : \tau_m[\Gamma]$, abbreviated $[t_1, \ldots, t_m] : \Gamma \to \Gamma'$. We take as arrows equivalence classes of context morphisms under the relation $[t_1, \ldots, t_n] = [s_1, \ldots, s_n]$ if and only if t_i is equivalent—as terms—to s_i, for all i. Contexts up to α-equivalence and context morphisms up to term-equivalence form a category.

Definition 5. *For a context Γ, let* $\mathrm{Form}_\Gamma \coloneqq \{\phi \mid \phi \text{ is a formula in context } \Gamma\}$*. The* syntactic hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MA}$ *sends objects Γ to*

$$P(\Gamma) \coloneqq \mathrm{Form}_\Gamma / \sim$$

where $\sim$ is the equivalence relation $\phi \sim \psi$ if and only if $\vdash_{\mathbf{TFOL+S}} \phi \supset\subset \psi$ $[\Gamma]$. The object $P(\Gamma)$ has a modal algebra structure induced by the logical connectives.

The syntactic hyperdoctrine sends arrows $[t_1, \ldots, t_m] : \Gamma \to \Gamma'$ to

$$P([t_1, \ldots, t_m]) : P(\Gamma') \to P(\Gamma),$$

defined by $P([t_1, \ldots, t_m])(\phi) \coloneqq \phi[t_1/y_1, \ldots, t_m/y_m]$.

Proposition 2. *The syntactic hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MA}$ is a modal hyperdoctrine.*

Proof. See Appendix B.

There is the obvious canonical interpretation (generic model) of $\mathbf{TFOL} + \mathbf{S}$ in the syntactic hyperdoctrine, about which we can say the following:

Proposition 3. *If ϕ $[\Gamma]$ is satisfied in the canonical interpretation in the syntactic hyperdoctrine then it is deducible in* $\mathbf{TFOL} + \mathbf{S}$*.*

From this it follows that if ϕ $[\Gamma]$ is satisfied in any interpretation in any modal hyperdoctrine, then it is deducible in $\mathbf{TFOL} + \mathbf{S}$.

4.3 Hyperdoctrinal Translation Theorem for TFOL + S4

Having categorical semantics for a logic allows us to investigate that logic using the structure of category theory; in the following, we compose an **S4** modal hyperdoctrine with a translation functor from modal to intuitionistic logic to get a hyperdoctrinal translation theorem. One direction of the Gödel-McKinsey-Tarski translation between modal and intuitionistic logic (see, e.g., [18]) can be expressed as the functor

$$\mathsf{Fix}_{\Box} : \mathbf{MA} \to \mathbf{HA}$$

sending a modal algebra A to a Heyting algebra on the set $\{a \in A \mid \Box a = a\}$, and an **MA**-homomorphism $h : A \to B$ to an **HA**-homomorphism

$$\mathsf{Fix}_{\Box}(h) : \mathsf{Fix}_{\Box}(A) \to \mathsf{Fix}_{\Box}(B).$$

For the functor to send modal algebras to Heyting algebras, the modal algebra must satisfy all the axioms in Definition 1, and thus the translation theorem only works for modal logics **S4** and stronger.

Proposition 4. *Let $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MA}$ be a modal hyperdoctrine. The functor*

$$P_{\mathsf{Fix}} := \mathsf{Fix}_{\Box} \circ P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$$

is an intuitionistic hyperdoctrine.

Proof. Firstly, we show that there are right and left adjoints, $\forall_{\pi}^{\Box}$ and $\exists_{\pi}^{\Box}$, to

$$P_{\mathsf{Fix}}(\pi) : P_{\mathsf{Fix}}(Y) \to P_{\mathsf{Fix}}(X \times Y),$$

where $\pi : X \times Y \to Y$ is a projection function in $\mathbf{C}$. Since P is a modal hyperdoctrine, there exist maps $\forall_{\pi}, \exists_{\pi} : P(X \times Y) \to P(Y)$ right and left adjoint to $P(\pi)$. We restrict these maps to the domain $P_{\mathsf{Fix}}(X \times Y)$ to define the right and left adjoints to $P_{\mathsf{Fix}}(\pi)$ as follows. For $\psi \in P_{\mathsf{Fix}}(X \times Y)$,

$$\forall_{\pi}^{\mathsf{Fix}}(\psi) := \mathsf{Fix}_{\Box}(\forall_{\pi}(\psi)) \quad \text{and} \quad \exists_{\pi}^{\mathsf{Fix}}(\psi) := \mathsf{Fix}_{\Box}(\exists_{\pi}(\psi)).$$

To show that $\forall_{\pi}^{\mathsf{Fix}}$ is right adjoint to $P_{\mathsf{Fix}}(\pi)$, let $\phi \in P_{\mathsf{Fix}}(Y)$ and suppose $P_{\mathsf{Fix}}(\pi)(\phi) \leq \psi$. Since $P_{\mathsf{Fix}}(\pi)$ is just the restriction $P(\pi)|_{P_{\mathsf{Fix}}(Y)}$, we have $P(\pi)(\phi) \leq \psi$, and since $P(\pi)$ is left adjoint to $\forall_{\pi}$, this means $\phi \leq \forall_{\pi}(\psi)$. But $\psi \in P_{\mathsf{Fix}}(X \times Y)$, so $\forall_{\pi}^{\mathsf{Fix}}(\psi) = \forall_{\pi}(\psi)$ and $\phi \leq \forall_{\pi}^{\mathsf{Fix}}(\psi)$. Since this argument is entirely reversible, the other direction of the bijection holds. A similar argument can be made to show $\exists_{\pi}^{\mathsf{Fix}}$ is left adjoint to $P_{\mathsf{Fix}}(\pi)$.

5 Higher-Order Modal Hyperdoctrine

From the hyperdoctrine perspective of "logic over type theory", moving from first-order logic to higher-order logic corresponds to adding more structure to the indexing category **C**. After specifying the higher-order syntax, we define a higher-order modal hyperdoctrine and prove soundness and completeness.

5.1 Higher-Order Modal Logic

We present a higher-order version of a typed modal system **S**, called **HoS**. This is achieved by two augmentations to the type structure of **TFOL** + **S**: to enable quantification over predicates, we add a special type of propositions to the signature; we also add rules for arrow and finite product types to the term calculus to give a simply typed λ-calculus. These changes follow [12] and [19].

Simply Typed $\boldsymbol{\lambda 1_\times}$-calculus. In addition to the basic types of our signature Σ we add *compound types* by including the usual type formation rules for arrow (exponent) types $\to$ and finite product types $1, \times$. We also add the usual introduction, elimination and computation rules for terms of these types: for arrow types, these are λ-abstraction, application, and β- and η-conversion; for finite product types, these are pairing, projection, and their conversion rules. Substitution is extended to these terms in the usual way (see [12, Section 2]).

Distinguished Type Prop. To be able to quantify over propositions as well as inhabitants of types σ, we add the distinguished type Prop to those listed in the signature. Like the other types, Prop has a list of variables $x, y, z, \ldots$.

On top of the signature, terms-in-context $M : \sigma\ [\Gamma]$ and formulae-in-context $\phi\ [\Gamma]$ are given the same inductive definition as in Sect. 3. Terms of type Prop (in context) are constructed as follows. For each relation symbol $R \subseteq \sigma_1, \ldots, \sigma_n$ in the signature, introduce a corresponding function symbol with codomain Prop:

$$R : \sigma_1, \ldots, \sigma_n \to \mathsf{Prop}.$$

Then for $M_1 : \sigma_1\ [\Gamma], \ldots, M_n : \sigma_n\ [\Gamma]$, there is a term $R(M_1, \ldots, M_n)$ of type Prop. Further terms of type Prop are constructed by the logical connectives:

$$\frac{\phi : \mathsf{Prop}\ [\Gamma] \quad \psi : \mathsf{Prop}\ [\Gamma]}{\phi * \psi : \mathsf{Prop}\ [\Gamma]} \text{ for } * \in \{\wedge, \vee, \supset\}$$

$$\frac{\phi : \mathsf{Prop}\ [\Gamma]}{*\phi : \mathsf{Prop}\ [\Gamma]} \text{ for } * \in \{\neg, \Box\} \qquad \frac{\phi : \mathsf{Prop}\ [x : \sigma, \Gamma]}{*_{x:\sigma}\phi : \mathsf{Prop}\ [\Gamma]} \text{ for } * \in \{\forall, \exists\}$$

Substitution over these terms is defined in the usual way (see [12] for full details).

On top of this term calculus, we still have the judgement $\vdash_{\mathbf{HoS}} \phi\ [\Gamma]$, saying that there is a derivation of $\phi\ [\Gamma]$ as governed by the first-order logic rules in Sect. 3. It remains to relate the notion of logical equivalence between formulae[3] to the notion of equality of terms of type Prop via the following rule:

$$\frac{\vdash_{\mathbf{HoS}} \phi \supset \psi\ [\Gamma] \quad \vdash_{\mathbf{HoS}} \psi \supset \phi\ [\Gamma]}{\phi = \psi : \mathsf{Prop}\ [\Gamma]}\ (\mathit{Prop}),$$

where $\phi = \psi : \mathsf{Prop}$ is judgemental (computational) equality of terms, that is, one term may be converted to the other following the rules of the λ-calculus. Propositions are now terms internal to the type theory.

[3] For convenience in the proofs to come, we express it as two separate conditionals rather than the biconditional $\supset\subset$.

5.2 Modal Tripos

Definition 6. *A* modal tripos, *or* higher-order modal hyperdoctrine, *is a modal hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MA}$ where the base category is cartesian closed and there is a* truth-value object *Ω in $\mathbf{C}$ with a natural isomorphism*

$$P(C) \simeq Hom_{\mathbf{C}}(C, \Omega)$$

Modal tripos semantics are given by the following definition.

Definition 7. *Fix a modal tripos $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MA}$. An interpretation $[\![\text{-}]\!]$ of* **HoS** *in P is given by the interpretation in Definition 3, augmented as follows.*

- *arrow and finite product types, $\sigma \to \tau$ and $1, \sigma \times \tau$, are interpreted by exponentiation $[\![\tau]\!]^{[\![\sigma]\!]}$ and categorical product $[\![\sigma]\!] \times [\![\tau]\!]$ in $\mathbf{C}$;*
- *the following cases are added to the inductively-defined interpretation of a term: λ-abstraction, λ-application, pairing and projections are interpreted by categorical transpose, evaluation, pairing and projection respectively in $\mathbf{C}$;*
- *the type Prop is assigned to the truth-value object Ω in $\mathbf{C}$, i.e. $[\![\mathsf{Prop}]\!] = \Omega$;*
- *a term $\phi : \mathsf{Prop}\ [\Gamma]$ is assigned to the arrow $[\![\phi]\!] : [\![\Gamma]\!] \to [\![\mathsf{Prop}]\!]$ in $\mathbf{C}$ that corresponds to $[\![\phi\ [\Gamma]]\!] \in P([\![\Gamma]\!])$ via the defining isomorphism of P.*

5.3 Soundness and Completeness

Proposition 5. *If $\phi\ [\Gamma]$ has a derivation in $\mathbf{HoE}_X$, then it is satisfied in any interpretation in any modal tripos.*

Proof. Fix a modal tripos P and an interpretation $[\![\text{-}]\!]$ in P. With the soundness of modal hyperdoctrine semantics established in Proposition 1, it remains to show that satisfaction of the *Prop* rule is preserved. Suppose $[\![\phi \supset \psi]\!]$ and $[\![\psi \supset \phi]\!]$ are true in $P([\![\Gamma]\!])$, and so we have

$$\top \leq [\![\phi]\!] \supset [\![\psi]\!] \text{ and } \top \leq [\![\psi]\!] \supset [\![\phi]\!].$$

By 2, this is equivalent to $[\![\phi]\!] \leq [\![\psi]\!]$ and $[\![\psi]\!] \leq [\![\phi]\!]$, from which it follows that $[\![\phi]\!] = [\![\psi]\!]$ in $P([\![\Gamma]\!])$. By the isomorphism in the definition of a modal tripos, $[\![\phi]\!], [\![\psi]\!] \in P([\![\Gamma]\!])$ correspond to arrows $[\![\phi]\!], [\![\psi]\!] : [\![\Gamma]\!] \to [\![\mathsf{Prop}]\!]$ in $\mathbf{C}$ that must be equal. These arrows are the interpretations of the terms $\phi : \mathsf{Prop}[\Gamma]$ and $\psi : \mathsf{Prop}[\Gamma]$ respectively, and so we have:

$$[\![\phi : \mathsf{Prop}[\Gamma]]\!] = [\![\psi : \mathsf{Prop}[\Gamma]]\!]$$

which is the same as $[\![\phi = \psi : \mathsf{Prop}\ [\Gamma]]\!]$.

Towards proving completeness, we are interested in the syntactic tripos of **HoS**, which is defined in the same way as the syntactic hyperdoctrine. Here we prove that it is in fact a modal tripos.

Proposition 6. *The syntactic hyperdoctrine defined in 5 is a modal tripos.*

Proof. The existence of finite products and exponentials in $\mathbf{C}$ is guaranteed by the existence of finite product types and function types in the type theory. To show the existence of a truth value object, we need a context up to α-equivalence satisfying the required isomorphism. Take $\Omega = x : \mathsf{Prop}$, noting that this is essentially the same as taking Prop itself when considering $x : \mathsf{Prop}$ as a (single variable) context up to α-equivalence. The required isomorphism then becomes

$$P(\Gamma) \simeq \mathrm{Hom}_{\mathbf{C}}(\Gamma, x : \mathsf{Prop}).$$

By the definition of the syntax, for every formula $\phi\ [\Gamma]$—built from atomic formulae $R(M_1, \ldots, M_n)\ [\Gamma]$ and the logical connectives—there is a corresponding term $\phi : \mathsf{Prop}\ [\Gamma]$—built in the same way from the logical connectives and atomic propositions $R(M_1, \ldots, M_n) : \mathsf{Prop}\ [\Gamma]$. We may consider the term $\phi : \mathsf{Prop}\ [\Gamma]$ as a context morphism in the base category of the modal tripos, that is, as a list of terms of length one, $[\phi] : \Gamma \to \mathsf{Prop}$. This gives the following isomorphism:

$$P(\Gamma) \simeq P(\Gamma, \mathsf{Prop}).$$

To show that this isomorphism is natural, for contexts $\Gamma = \sigma_1, \ldots, \sigma_n$ and $\Gamma' = \tau_1, \ldots, \tau_m$, for any morphism $[t_1, \ldots, t_m] : \Gamma \to \Gamma'$ in the base category, where $t_i : \tau_i\ [\Gamma]$, we require that the following square commutes:

$$\begin{tikzcd}
P(\Gamma') \arrow[r, "\mathrm{PaF}'_{\Gamma}"] \arrow[d, "{P([t_1,\ldots,t_m])}"'] & \mathrm{Hom}_{\mathbf{C}}(\Gamma', \mathsf{Prop}) \arrow[d, "{\mathrm{Hom}_{\mathbf{C}}([t_1,\ldots,t_m], x:\mathsf{Prop})}"] \\
P(\Gamma) \arrow[r, "\mathrm{PaF}_{\Gamma}"'] & \mathrm{Hom}_{\mathbf{C}}(\Gamma, \mathsf{Prop})
\end{tikzcd}$$

where PaF ("Propositions as functions") denotes the isomorphism. This is given by the calculation:

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{C}}([t_1, \ldots, t_m], x : \mathsf{Prop}) \circ \mathrm{PaF}_{\Gamma}(\phi\ [\Gamma']) \\
&= \mathrm{Hom}_{\mathbf{C}}([t_1, \ldots, t_m], x : \mathsf{Prop})(\phi : \mathsf{Prop}\ [\Gamma']) \\
&= \phi[t_1/x_1, \ldots, t_m/x_m] : \mathsf{Prop}\ [\Gamma] \\
&= \mathrm{PaF}_{\Gamma}(\phi[t_1/x_1, \ldots, t_m/x_m][\Gamma]) \\
&= \mathrm{PaF}_{\Gamma} \circ P([t_1, \ldots, t_m])(\phi[\Gamma']).
\end{aligned}$$

It is straightforward to see that if $\phi\ [\Gamma]$ is valid in the canonical interpretation in the syntactic tripos, then it is provable in **HoS**. The standard counter-model argument then immediately gives completeness. Combined with soundness, we obtain the following theorem.

Theorem 1. *$\phi\ [\Gamma]$ is provable in* **HoS** *iff it is valid in any interpretation in any modal tripos.*

6 Conclusion

We have established both first-order and higher-order completeness for non-normal modal logics via hyperdoctrine semantics; we have also shown a hyperdoctrinal translation theorem for normal modal hyperdoctrines. The straightforward nature of these results is demonstrative of the power of hyperdoctrine.

Coalgebraic logic has been highly successful for a unified treatment of various propositional modal logics [4,17], and in future work, we plan to apply coalgebraic logic, especially duality-theoretical results such as in [14], to construct models of modal hyperdoctrines; the predicate functors of Stone-type dual adjunctions often form hyperdoctrines. More ambitiously, we plan to extend coalgebraic modal logic to first-order and higher-order settings via hyperdoctrine semantics.

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A Soundness of Modal Hyperdoctrine Semantics

Proof. Fix a modal hyperdoctrine P and an interpretation $[\![\text{-}]\!]$ in P. The proof is by induction on the derivation of $\phi\ [\Gamma]$, which amounts to checking that all axiom schemata are satisfied and that all rules preserve satisfaction.

For the propositional fragment, beginning with rule RE, suppose $[\![\phi \supset\subset \psi]\!]$ is true in $P([\![\Gamma]\!])$. Expanding the abbreviation $\supset\subset$ and taking the interpretation of the connectives as in Definition 3, we have:

$$[\![\phi \supset \psi \wedge \psi \supset \phi]\!] = [\![\phi]\!] \supset [\![\psi]\!] \wedge [\![\psi]\!] \supset [\![\phi]\!].$$

It is a theorem in a Boolean algebra that the right-hand side implies $[\![\phi]\!] = [\![\psi]\!]$. Therefore, $\Box[\![\phi]\!] = \Box[\![\psi]\!]$, which is the interpretation of the formula $\Box\phi \supset\subset \Box\psi\ [\Gamma]$, and so the rule RE preserves satisfaction. The rule MP may be checked in a similar way.

For the modal axiom schemata, satisfaction of schema E corresponds to the definition of the $\Diamond$ operator in a modal algebra, so $\Diamond[\![\phi]\!] = \neg\Box\neg[\![\phi]\!]$ in $P([\![\Gamma]\!])$. The interpretation of schema M is:

$$[\![\Box(\phi \wedge \psi) \supset (\Box\phi \wedge \Box\psi)]\!] = \Box([\![\phi]\!] \wedge [\![\psi]\!]) \supset (\Box[\![\phi]\!] \wedge \Box[\![\psi]\!]),$$

so M is satisfied in the interpretation if $\Box([\![\phi]\!] \wedge [\![\psi]\!]) \leq (\Box[\![\phi]\!] \wedge \Box[\![\psi]\!])$, by the equivalent condition for satisfaction of an implication established in 2. This corresponds clearly to condition M_A on the operator.

In a similar way, we can show that schema N corresponds to condition N_A and C corresponds to C_A. It is also clear that we may add more axiom schemata to $\mathbf{TFOL} + \mathbf{E}_X$ to get a system $\mathbf{TFOL} + \mathbf{S}$, and that these schemata are satisfied in the interpretation if we add corresponding conditions on the modal operator in the algebra. Satisfaction of the axiom schemata for the non-modal part of the proportional logic may be verified in the same way.

For the first-order fragment, the axiom schema $\forall$-*Elimination* is satisfied if and only if the interpretation $[\![(\forall x\phi)[x_1,\ldots,x_n] \supset \phi]\!]$ is true in $P([\![\sigma]\!] \times [\![\Gamma]\!])$. By 2, we can do this by showing

$$[\![(\forall x\phi)[x_1,\ldots,x_n]]\!] \leq [\![\phi]\!].$$

The logical expression on the left-hand side, $(\forall x\phi)[x_1,\ldots,x_n]$, is a formula in context $[x:\sigma,\Gamma]$, but which does not contain x, and so corresponds to weakening of the context. By the semantics of substitution, we have:

$$[\![(\forall x\phi)[x_1,\ldots,x_n]\ [x:\sigma,\Gamma]]\!] = P(\pi)([\![\forall x\phi\ [\Gamma]]\!]),$$

and by the interpretation of the universal quantifier,

$$P(\pi)([\![\forall x\phi\ [\Gamma]]\!]) = P(\pi)(\forall_\pi([\![\phi\ [x:\sigma,\Gamma]]\!])).$$

This turns the desired statement into another form of the adjointness condition for universal quantification, that is, the counit characterisation:

$$P(\pi)(\forall_\pi([\![\phi\ [x:\sigma,\Gamma]]\!])) \leq [\![\phi\ [x:\sigma,\Gamma]]\!].$$

To show that the $\forall$-*Introduction* rule preserves satisfaction, suppose

$$[\![\phi[x_1,\ldots,x_n] \supset \psi]\!]$$

is true in $P([\![\sigma]\!] \times [\![\Gamma]\!])$, or equivalently, $[\![\phi[x_1,\ldots,x_n]]\!] \leq [\![\psi]\!]$. Then we need to show that $[\![\phi \supset \forall x\psi]\!]$ is true in $P([\![\Gamma]\!])$, or equivalently, $[\![\phi]\!] \leq \forall_\sigma[\![\psi]\!]$. This logical rule directly translates to one direction of the adjointness correspondence when we observe that the formula $\phi[x_1,\ldots,x_n]\ [x:\sigma,\Gamma]$ is weakening of the formula $\phi[\Gamma]$. By the interpretation of substitution,

$$[\![\phi[x_1,\ldots,x_n]\ [x:\sigma,\Gamma]]\!] = P(\pi)([\![\phi\ [\Gamma]]\!]).$$

But if $P(\pi)([\![\phi]\!]) \leq [\![\psi]\!]$ holds, then by the adjointness condition for universal quantification, $[\![\phi]\!] \leq \forall_\sigma[\![\psi]\!]$ as required.

For the *Instantiation* rule, suppose $\top \leq [\![\phi]\!]$ in $P([\![\Gamma]\!])$. Applying the (order-preserving) modal algebra homomorphism

$$P(\langle[\![M_1:\sigma_1\ [\Gamma']]\!],\ldots,[\![M_n:\sigma_n\ [\Gamma']]\!]\rangle) : P([\![\Gamma]\!]) \to P([\![\Gamma']\!])$$

to both sides, we get:

$$\begin{aligned} &P(\langle[\![M_1:\sigma_1\ [\Gamma']]\!],\ldots,[\![M_n:\sigma_n\ [\Gamma']]\!]\rangle)(\top) \\ &\qquad\qquad \leq P(\langle[\![M_1:\sigma_1\ [\Gamma']]\!],\ldots,[\![M_n:\sigma_n\ [\Gamma']]\!]\rangle)([\![\phi\ [\Gamma]]\!]). \end{aligned}$$

Since modal algebra homomorphisms preserve $\top$, and by the semantics of substitution, we have $\top \leq [\![\phi[\vec{M}/\vec{x}]]\!]$ in $P([\![\Gamma']\!])$.

B Completeness of Modal Hyperdoctrine Semantics

Proof. Firstly, the base category **C** has finite products: for $\Gamma = \sigma_1, \ldots, \sigma_n$ and $\Gamma' = \tau_1, \ldots, \tau_m$, define $\Gamma \times \Gamma'$ as $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n$. We then have, as an associated projection,

$$[y_1, \ldots, y_m] : \Gamma \times \Gamma' \to \Gamma',$$

where the y_i are variables $y_i : \tau_i \ [\Gamma, \Gamma']$. The other projection is defined similarly, and it is straighforward to show that this gives a categorical product in **C**.

Next, we check that the codomain of P is in fact the category of modal algebras and structure-preserving homomorphisms. For a context Γ in **C**, $P(\Gamma)$ forms a modal algebra with operations induced in the expected way by the logical connectives. Considering only the non-modal fragment of the logic, $P([t_1, \ldots, t_n])$ is a Boolean algebra homomorphism since substitution commutes with all the non-modal logical operations. To extend this to a modal algebra homomorphism, we require that $P([t_1, \ldots, t_n])$ preserves the modal operator $\Box$ and any extra conditions placed on $\Box$. This follows from the fact that $P([t_1, \ldots, t_n])$ performs substitution into a formula, and the syntax specifies that $\Box$ commutes with substitution.

Proceeding to the quantifier structure, universal quantification is given by a right adjoint to $P(\pi) : P(\Gamma') \to P(\Gamma \times \Gamma')$, where $\pi : \Gamma \times \Gamma' \to \Gamma'$ is the second projection in **C**. Let ψ be a formula in $P(\Gamma \times \Gamma')$; since the following arguments respect equivalence, we will identify ψ with the equivalence class to which it belongs. Define $\forall_\pi : P(\Gamma \times \Gamma') \to P(\Gamma')$ by

$$\forall_\pi(\psi) \coloneqq \forall x_1 \ldots \forall x_n \psi,$$

with the formula on the right hand side denoting the corresponding equivalence class.

Suppose $\phi \in P(\Gamma')$; to show that $\forall_\pi$ is the right adjoint of $P(\pi)$ means showing $P(\pi)(\phi) \leq \psi$ in $P(\Gamma \times \Gamma')$ if and only if $\phi \leq \forall x_1 \ldots \forall x_n \psi$ in $P(\Gamma')$. For the first direction, assume $P(\pi)(\phi) \leq \psi$ in $P(\Gamma \times \Gamma')$. Since $P(\pi)(\phi)$ corresponds to weakening of the context of $\phi \ [\Gamma]$ to $\phi[y_1, \ldots, y_m] \ [\Gamma, \Gamma']$, we have $\phi[y_1, \ldots, y_m] \leq \psi$ in $P(\Gamma \times \Gamma')$. The partial order in $P(\Gamma \times \Gamma')$ is induced by its lattice structure, so the above ordering corresponds to the equation $\phi[y_1, \ldots, y_m] \wedge \psi = \phi[y_1, \ldots, y_m]$. By the definition of the syntactic hyperdoctrine, we can make the following derivability statement:

$$\vdash_{\mathbf{TFOL+S}} \phi[y_1, \ldots, y_m] \wedge \psi \supset\subset \phi[y_1, \ldots, y_m] \ [\Gamma, \Gamma']$$

from which it follows that

$$\vdash_{\mathbf{TFOL+S}} \phi[y_1, \ldots, y_m] \supset \psi \ [\Gamma, \Gamma'].$$

Repeated application of $\forall$-*Introduction* gives

$$\vdash_{\mathbf{TFOL+S}} \phi \supset \forall x_1 \ldots \forall x_n \psi \quad [\Gamma'],$$

from which it follows that

$$\vdash_{\mathbf{TFOL+S}} \phi \wedge \forall x_1 \ldots \forall x_n \psi \supset\subset \phi \quad [\Gamma'].$$

Translating back to the modal algebra, this means $\phi \wedge \forall x_1 \ldots \forall x_n \psi = \phi$ in $P(\Gamma')$, and so $\phi \leq \forall x_1 \ldots \forall x_n \psi$ in $P(\Gamma')$, as required.

For the other direction, assume $\phi \leq \forall x_1 \ldots \forall x_n \psi$ in $P(\Gamma')$. Using the same reasoning as before to translate from a statement in the modal algebra to one in the logic, we have

$$\vdash_{\mathbf{TFOL+S}} \phi \wedge \forall x_1 \ldots \forall x_n \psi \supset\subset \phi \quad [\Gamma'],$$

from which it follows:

$$\vdash_{\mathbf{TFOL+S}} \phi \supset \forall x_1 \ldots \forall x_n \psi \quad [\Gamma'].$$

Applying the *Instantiation* rule to weaken the context gives

$$\vdash_{\mathbf{TFOL+S}} (\phi \supset \forall x_1 \ldots \forall x_n \psi)[y_1, \ldots, y_m] \quad [x_1 : \sigma_1, \Gamma'],$$

where we substitute for the variables $y_i : \tau_i$ $[\Gamma']$ variables $y_i : \tau_i$ $[x_1 : \sigma_1, \Gamma']$. Since substitution commutes with $\supset$, we have:

$$\vdash_{\mathbf{TFOL+S}} \phi[y_1, \ldots, y_m] \supset (\forall x_1 \ldots \forall x_n \psi)[y_1, \ldots, y_m] \quad [x_1 : \sigma, \Gamma'] \tag{4}$$

We will prove $\phi[y_1, \ldots, y_m] \supset \forall x_2 \ldots \forall x_n \psi$ $[x_1 : \sigma_1, \Gamma']$ using the deduction theorem. Assume

$$\vdash_{\mathbf{TFOL+S}} \phi[y_1, \ldots, y_m] \; [x_1 : \sigma_1, \Gamma'], \tag{5}$$

then applying rule MP (modus ponens) to 5 and 4 gives:

$$\vdash_{\mathbf{TFOL+S}} (\forall x_1 \ldots \forall x_n \psi)[y_1, \ldots, y_m] \quad [x_1 : \sigma_1, \Gamma']. \tag{6}$$

The following is an instance of the $\forall$-*Elimination* schema:

$$\vdash_{\mathbf{TFOL+S}} (\forall x_1 \forall x_2 \ldots \forall x_n \psi)[y_1, \ldots, y_m] \supset \forall x_2 \ldots \forall x_n \psi \quad [x_1 : \sigma_1, \Gamma']. \tag{7}$$

Applying modus ponens to 6 and 7:

$$\vdash_{\mathbf{TFOL+S}} \forall x_2 \ldots \forall x_n \psi \; [x_1 : \sigma_1, \Gamma'].$$

Since this follows from the assumption that $\phi[y_1, \ldots, y_m]$ $[x_1 : \sigma_1, \Gamma']$ is derivable, we have

$$\vdash_{\mathbf{TFOL+S}} \phi[y_1, \ldots, y_m] \supset \forall x_2 \ldots \forall x_n \psi \; [x_1 : \sigma_1, \Gamma']$$

by the deduction theorem. Repeating this argument, we get

$$\vdash_{\mathbf{TFOL+S}} \phi[y_1, \ldots, y_m] \supset \psi \; [\Gamma, \Gamma'],$$

and translating this back into a statement in the modal algebra $P(\Gamma')$, we have $P(\pi)\phi \leq \psi$.

To show that the corresponding Beck-Chevalley condition is satisfied, let $\Gamma'' = v_1 : \mu_1, \ldots, v_l : \mu_l$ be a context up to α-equivalence. Then for every context morphism $[s_1, \ldots, s_m] : \Gamma'' \to \Gamma'$ with $s_i : \tau_i\ [\Gamma'']$ the following diagram must commute:

$$\begin{tikzcd}
P(\Gamma \times \Gamma') \arrow[r, "\forall_\pi"] \arrow[d, "P(\mathsf{id}_\Gamma \times [s_1,\ldots,s_m])"'] & P(\Gamma') \arrow[d, "P([s_1,\ldots,s_m])"] \\
P(\Gamma \times \Gamma'') \arrow[r, "\forall_{\pi'}"'] & P(\Gamma'')
\end{tikzcd}$$

where $\pi' : \Gamma \times \Gamma'' \to \Gamma''$ is a projection. Since we specified in the term calculus that the quantifiers commute with substitution, we can make the following argument, for $\psi \in P(\Gamma \times \Gamma')$:

$$\begin{aligned}
P([s_1, \ldots, s_m]) \circ \forall_\pi(\psi) &= P([s_1, \ldots, s_m])(\forall x_1 \ldots \forall x_n \psi) \\
&= (\forall x_1 \ldots \forall x_n \psi)[s_1/y_1, \ldots, s_m/y_m] \\
&= \forall x_1 \ldots \forall x_n (\psi[s_1/y_1, \ldots, s_m/y_m]) \\
&= \forall_{\pi'}(\psi[s_1/y_1, \ldots, s_m/y_m]) \\
&= \forall_{\pi'} \circ P(1_\Gamma \times [s_1, \ldots, s_m])(\psi)
\end{aligned}$$

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