Homotopy Type Theory: Models and Modalities

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Overview

Main characters

- the "universe of fibrations" in models of HoTT
  (e.g. Licata, Orton, Pitts and Spitters, 2018)
- the $\Box$-modality from S4 modal logic
  (Lewis 1932, Gödel 1933)

Plan

1. HoTT and its models
   ↳ types as fibrations
2. Working with models of HoTT
   ↳ two approaches
   ↳ the universe of fibrations
3. Modalities in HoTT
4. Relating two descriptions of the universe of fibrations

Background

Equality in type theory

- Formation rule

$$\frac{x, y : A}{x =_A y \text{ is a type}}$$

can be iterated

$$\frac{p, q : x =_A y}{p =_{x =_A y} q \text{ is a type}}$$

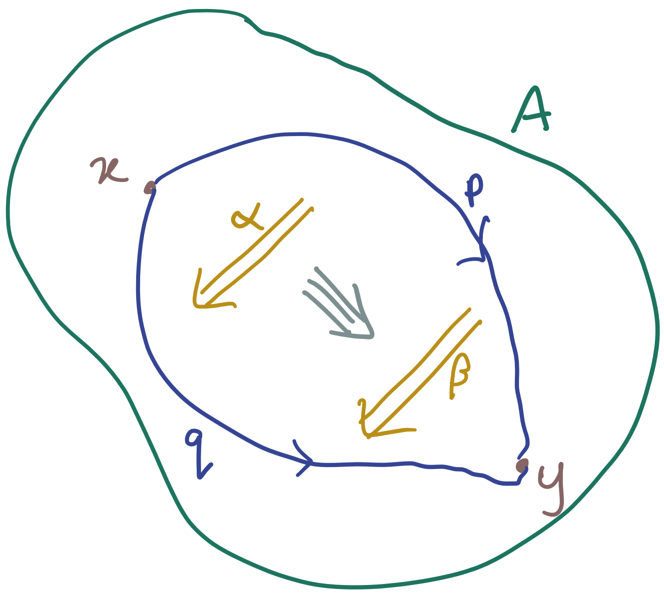
- and again

$$\frac{\alpha, \beta : p =_{x =_A y} q}{\alpha =_{p =_{x =_A y} q} \beta \text{ is a type}} \quad \cdots$$

How do we make sense of this?

Intuition

$\alpha =_{p =_{x =_A y} q} \beta$ is a type



A is a space
x is a point
p is a path
α is a homotopy ...

Homotopical interpretation of type theory

- the development of homotopical models of equality types (and the other type constructors)

(Voevodsky 2006, Gambino and Garner 2007, Awodey and Warren 2008)

- Types are spaces with "higher-dimensional structure", modelled by "fibrations".

- Homotopy type theory = Martin-Löf type theory
  + univalence axiom
  + homotopy levels
  + higher inductive types

- comes from studying models, and we continue to study models

Exploring models of HoTT

- original model of HoTT isn't constructive
- constructive models have subsequently been developed
- want to study their pros and cons

involves carrying around algebraic structure on certain maps

(Coquand 2015)

work in a type-theoretic language

Example: a "trivial fibration structure" on ...

① (category-theoretic)

∴ p is a choice of diagonal fillers $j(m,u,v)$

$$\begin{tikzcd} S \arrow[r, "u"] \arrow[d, hook, "m"'] & A \arrow[d, "p"] \\ T \arrow[r, "v"'] \arrow[ur] & X \end{tikzcd}$$

for all $m \in \mathrm{Cof}$ such that

$$\begin{tikzcd} t^*(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r, "u"] \arrow[d] & A \arrow[d, "p"] \\ T' \arrow[r, "t"'] \arrow[urr] & T \arrow[r, "v"'] \arrow[ur] & X \end{tikzcd}$$

for all $m \in \mathrm{Cof}$, for all $t : T' \to T$.

② (type-theoretic)

... $\alpha : X \to \mathcal{U}$ is an element

$$t : \mathrm{TFib}(\alpha)$$

where

$$\mathrm{TFib}(\alpha) = \prod_{\varphi : \Phi} \prod_{v : \alpha^{\{\varphi\}}} \sum_{a : \alpha} v = \lambda(a)$$

Why can we do this?

Ingredients of a type theory		a presheaf category $\hat{\mathcal{C}}$
contexts Γ	$\rightsquigarrow$	objects Γ
type judgements $\Gamma \vdash \alpha$ type $\Leftrightarrow$ context extensions $\Gamma.\alpha \vdash$	$\rightsquigarrow$	special class of maps $\Gamma.\alpha \xrightarrow{\rho} \Gamma$
term judgements $\Gamma \vdash a : \alpha$	$\rightsquigarrow$	sections $\begin{tikzcd} \Gamma.\alpha \arrow[d, "\rho"] \\ \Gamma \arrow[u, bend left, "a"] \end{tikzcd}$

- Going the other way, from our category we can define a type theory

Ingredients of a type theory		a presheaf category $\hat{\mathcal{C}}$
contexts Γ	$\leftsquigarrow$	objects Γ
type judgements $\Gamma \vdash \alpha$ type & context extensions $\Gamma.\alpha \vdash$	$\leftsquigarrow$	special class of maps $\Gamma.\alpha \xrightarrow{p} \Gamma$
term judgements $\Gamma \vdash a : \alpha$	$\leftsquigarrow$	sections $\Gamma \xrightarrow{a} \Gamma.\alpha$, $\Gamma.\alpha \xrightarrow{p} \Gamma$

- we obtain an "internal type theory" of $\hat{\mathcal{C}}$

Reasoning in an internal type theory

- Internal approach to cubical models (Orton and Pitts, 2016) allows you to work with types rather than diagrams

  ⇝ simpler descriptions
  ⇝ can be used to find new models

- Diagrammatic and type-theoretic descriptions can be related by

  i) directly unfolding the interpretation of the internal judgements

  OR

  ii) quasi-mechanically, via the technique of "Kripke-Joyal forcing"
  (Awodey, Gambino and Hazratpour, 2024)

The universe of fibrations

- Recall: types and their higher-dimensional structure are interpreted by fibrations
- "type universes" allow you to treat types as terms

$$A : \mathcal{U}$$

- when studying models of HoTT, not all types are fibrant
- we want a universe of fibrations / fibrant types

$$A : \mathcal{U}_{Fib}$$

to "package up" a type and its fibration structure.

the universe of fibrations :

① (category-theoretic)

$$\begin{tikzcd} \mathrm{Fib}^*(\mathrm{id}) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fill}(\mathrm{id} \circ -)_I \arrow[d] \\ \mathcal{U} \arrow[r, "\eta"'] & (\mathcal{U}^I)_I \end{tikzcd}$$

② (type-theoretic)

impossible!

Problem: the internal language can't talk about part of the construction

Solution: extend the internal language with a modality

(Licata, Orton, Pitts and Spitters (LOPS) 2018)

Modal type theory for internal universes

(Licata, Orton, Pitts and Spitters (LOPS) 2018)

- for a type-theoretic version of the universe of fibrations, we need a distinction between "global" and "local" variables
- use a modal "dual-context" type theory

$$\Delta \mid \Gamma \vdash a : A$$

global/modal variables $x : \square M$

local/regular variables $y : N$

- global elements can be used locally, but cannot depend on local variables
- "crisp type theory"

The problem

- There is a type-theoretic version of the universe of fibrations (LOPS 2018) using crisp type theory

Recall:

Diagrammatic and type-theoretic descriptions can be related by

i) directly unfolding the interpretation of the internal judgements

OR

ii) quasi-mechanically, via the technique of "Kripke-Joyal forcing"

⚠ we don't have either of these for crisp type theory

Internal crisp type theory?

A presheaf category $\widehat{\mathbb{C}}$ idempotent comonad $\flat$		ingredients of crisp type theory
?	$\rightsquigarrow$	dual-context $\Delta \mid \Gamma$
?	$\rightsquigarrow$	type $\Delta \mid \Gamma \vdash \alpha$ type context extension $\Delta \mid \Gamma, x{:}\alpha \vdash$
?	$\rightsquigarrow$	term-in-context $\Delta \mid \Gamma \vdash a : \alpha$

+ two kinds of context extension...

Internal crisp type theory

Our approach

1) zoom out - how can we model the features of a dual-context type theory?

   e.g. • context dependence of $\Delta \mid \Gamma$

   • two kinds of context extension

2) zoom back in - how does $\hat{\mathbb{C}}$, $\flat$ admit such a model?

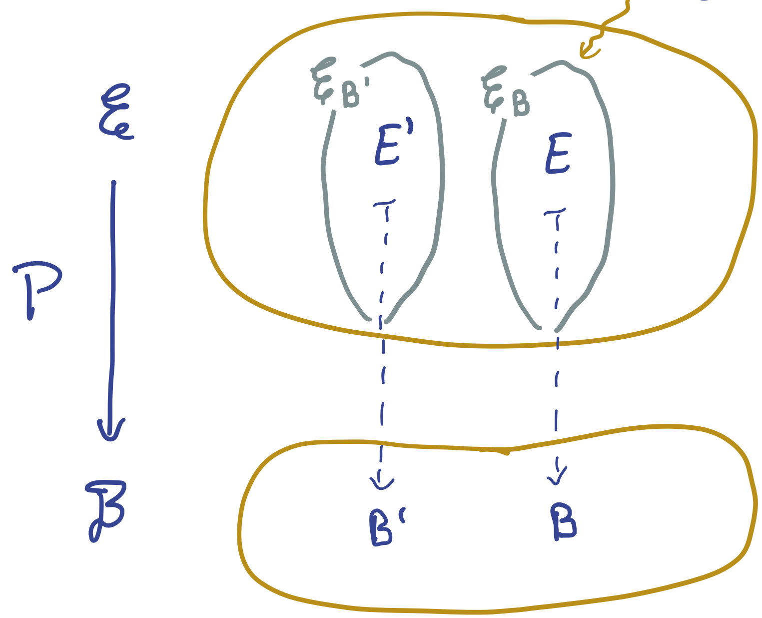
3) use this understanding to extract an internal crisp type theory

4) carry out the universe construction in this internal language, taking its interpretation to relate it to the diagrammatic version.

Modelling dual-context type theory

For a context $\Delta \mid \Gamma$, want to capture the dependency of Γ on Δ.

$\mathcal{E}_B$ is the "fibre category" over B



$E \in \mathcal{E}_B \quad \leftrightsquigarrow \quad \Delta \mid \Gamma$

$B \in \mathcal{B} \quad \leftrightsquigarrow \quad \Delta$

Fibred natural model of dual-context type theory

Idea Equip

(i) the base category, and
(ii) each fibre

with the structure to model a type theory.

e.g. Awodey's "natural models" (2016)

These structures should be related, i.e.

$$\Delta \vdash_{\mathcal{B}} \sigma \text{ type} \quad \text{“}=\text{”} \quad \Delta \mid \cdot \vdash_{\mathcal{E}} \sigma \text{ type}$$

⇝ "Fibred natural models of dual-context type theory" given by a functor $p : \mathcal{E} \to \mathcal{B}$ + axioms.

Zooming back in

Recall the intended models are categories with idempotent comonads (e.g. $\hat{\mathbb{C}}$, $\flat$)

Theorem $\hat{\mathbb{C}}, \flat$ gives rise to a fibred natural model.

$$\begin{tikzcd}
\mathcal{E} := \hat{\mathbb{C}} \downarrow \hat{\mathbb{C}}_\flat \arrow[d, "\mathrm{cod}"] \\
\mathcal{B} := \hat{\mathbb{C}}_\flat
\end{tikzcd}$$

full subcategory of $X \in \hat{\mathbb{C}}$ with $\flat X = X$

Internal crisp type theory

the comma category $\hat{\mathcal{C}} \downarrow \hat{\mathcal{C}}_b$

ingredients of crisp type theory

objects
$$\begin{tikzcd} \Theta \arrow[d, "\Delta|\Gamma"] \\ \Delta \end{tikzcd}$$
$\rightsquigarrow$ contexts $\Delta | \Gamma$

"vertical" small maps
$$\begin{tikzcd} & \Theta.\alpha \arrow[dl, "p"'] \arrow[d] \\ \Theta \arrow[d] & \Delta \arrow[dl, equal] \\ \Delta & \end{tikzcd}$$
$p \in \mathcal{S}$

$\rightsquigarrow$ types $\Delta | \Gamma \vdash \alpha$ type

context extension $\Delta | \Gamma, x : \alpha \vdash$

"horizontal" small maps
$$\begin{tikzcd} \Theta * \alpha \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta \arrow[d] \\ \Delta.\alpha \arrow[r, tail] & \Delta \end{tikzcd}$$
$\rightsquigarrow$ crisp context extension

$$\frac{\Delta | \cdot \vdash \alpha \ \text{type} \qquad \Delta | \Gamma \vdash}{\Delta, x :: \alpha \,|\, \Gamma \vdash}$$

Summary

- models of HoTT benefit from being explored using a type-theoretic language
- for the universe of fibrations, this language must have a modality ("crisp type theory")
- this is a complication for relating the diagrammatic and type-theoretic versions ...
- ... but this work fills the gap

Thank you!